THÈSE

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par **Luis Fredes**

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Quelques modèles à l'interface des probabilités et de la combinatoire : processus de particules et cartes.

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Devant la commission d'examen composée de :

Marie Albenque	Chargée de recherche, CNRS, École polytechnique	Examinatrice
Mireille Bousquet-Mélou	Directrice de recherche, CNRS, LaBRI	Examinatrice
Nicolas Broutin	Professeur, LPSM-Sorbonne Université	President du jury
Jean-François Le Gall	Professeur, Paris-sud Orsay	Examinateur
Jean-François Marckert	Directeur de recherche, CNRS, LaBRI	Directeur
Gregory Miermont	Professeur, ENS Lyon	Rapporteur
Peter Mörters	Professeur, Université de Cologne	Rapporteur

– 2019 –

Résumé Cette thèse se compose de plusieurs travaux portant sur deux branches de la théorie des probabilités: processus de particules et cartes planaires aléatoires.

Un premier travail concerne les aspects algébriques des mesures invariantes des processus de particules. Nous obtenons des conditions nécessaires et suffisantes sous lesquelles un processus de particules en temps continu avec espace d'états local discret possède une mesure invariante simple.

Dans un deuxième travail nous étudions un modèle "biologique" de coexistence de 2 espèces en compétition sur un espace partagé, et soumis à des épidémies modélisées par un modèle probabiliste appelé "feux de forêts". Notre résultat principal montre que pour deux espèces, il existe des régions explicites de paramètres pour lesquelles une espèce domine ou les deux espèces coexistent. Il s'agit d'un des premiers modèles pour lesquels la coexistence d'espèces sur le long terme est prouvée.

Les troisièmes et quatrièmes travaux. portent sur les cartes planaires décorées par des arbres. Dans le troisième nous présentons une bijection entre l'ensemble des cartes décorées par des arbres et le produit Cartésien entre l'ensemble des arbres planaires et l'ensemble de cartes à bord simple. Nous obtenons quelques formules de comptage et quelques outils pour l'étude de cartes aléatoires décorées par un arbre. Le quatrième travail montre que les triangulations et quadrangulations aléatoires uniformes avec f faces, bord simple de taille p et décorées par un arbre avec a arêtes, convergent en loi pour la topologie locale vers différentes limites, dépendant du comportement fini ou infini de la limite de f, p et a.

Mots-clés Processus de particules, lois invariantes, coexistence, extinction, comptage bijective de cartes, cartes aleatoires, limite locale.

Title Some models at the interface of probability and combinatorics : particle systems and maps.

Abstract This thesis consists in several works exploring some models belonging to two branches of probability theory: interacting particle systems and random planar maps.

A first work concerns algebraic aspects of interacting particle systems invariant measures. We obtain some necessary and sufficient conditions for some continuous time particle systems with discrete local state space, to have a simple invariant measure.

In a second work we investigate the effect on survival and coexistence of introducing forest fire epidemics to a certain two-species spatial competition model. Our main results show that, for the two-type model, there are explicit parameter regions where either one species dominates or there is coexistence; contrary to the same model without forest fires, for which the fittest species always dominates.

The third and fourth works are related to tree-decorated planar maps. In the third work we present a bijection between the set of tree-decorated maps and the Cartesian product between the set of trees and the set of maps with a simple boundary. We obtain some counting results and some tools to study random decorated map models. In the fourth work we prove that uniform tree-decorated triangulations and quadrangulations with f faces, boundary of length p and decorated by a tree of size a converge weakly for the local topology to different limits, depending on the finite or infinite behavior of f, p and a.

Keywords Interacting particle systems, invariant measures, coexistence, extinction, bijective map counting, random maps, limit local.

Laboratoire d'accueil LaBRI. 351, cours de la Libération, F-33405 Talence cedex, France

Résumé étendu en français (**it can be skipped to a much detailed introduction in english**).

A. Mesures invariantes de processus de particules discrètes : aspects algébriques.

Travail en commun avec J.F. Marckert.

Dans ce travail [50] nous nous intéressons à une classe spéciale de processus de particules (IPS) : il s'agit de processus de Markov en temps continus $(\eta_t)_{t\in\mathbb{R}^+}$ prenant leurs valeurs dans l'ensemble des colorations d'un graphe $G = (V, E)$, c'est-à-dire

$$\eta_t = (\eta_t(u), u \in V) \in E_\kappa^V,$$

où l'ensemble des couleurs est $E_\kappa := \{0, 1, \ldots, \kappa - 1\}$ pour un $\kappa \in \{\infty, 2, 3, \ldots\}$.

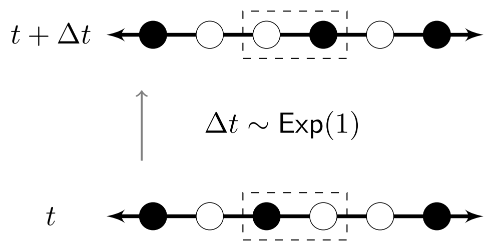
Les graphes considérés ici seront le réseau $\mathbb{Z}^d$ pour $d \geq 1$ ou $\mathbb{Z}/n\mathbb{Z}$ avec $n \geq 1$ ou le segment $[\![0, n]\!]$, pour $n \geq 1$.

Dans le cas de $\mathbb{Z}$ et $\mathbb{Z}/n\mathbb{Z}$, la dynamique du processus est définie par une matrice de taux de saut

$$\mathsf{T} = [\mathsf{T}_{[u|v]}]_{u,v\in E_\kappa^L},$$

oú la quantité $L \geq 2$ est appelée la portée de l'interaction (taille du voisinage d'influence). L'entrée $\mathsf{T}_{[u|v]}$ est la vitesse à laquelle chaque sous-mot de η égal à u (de longueur L) est transformé en v (pour u et v mots quelconques de longueur L). Voici un exemple.

Exemple (TASEP) Considérons la ligne $\mathbb{Z}$ vue comme un graphe $G = (\mathbb{Z}, E)$ avec ensemble d'arêtes $E = \{\{x, x + 1\}, x \in \mathbb{Z}\}$. Dans ce processus de particules, chaque particule essaie de sauter à droite au taux 1 et le saut ne devient effectif que si le site d'arrivée est vacant. Les sites noirs (blancs) représentent la présence (absence) d'une particule. La matrice de taux de saut est donnée par



$$\mathsf{T}_{[w|w']} = \begin{cases} 1 & \text{si } w = (\bullet, \circ) \text{ et } w' = (\circ, \bullet) \\ 0 & \text{sinon} \end{cases}$$

Cela signifie que chaque paire de sites dont les couleurs sont $(\bullet, \circ)$ attend un temps aléatoire exponentiel de taux 1 pour passer à $(\circ, \bullet)$ (les couleurs des autres sites restent les mêmes), après les horloges exponentielles sont redémarrées.

Définition : *Une distribution μ sur E_κ^V est dite* ***invariante*** *pour l'IPS de matrice de taux de saut* T *(ou simplement invariante par* T*) si $\eta_t \stackrel{(d)}{=} \eta_0 \stackrel{(d)}{=} \mu$ pour tout $t \geq 0$, où $\stackrel{(d)}{=}$ désigne l'égalité en distribution.*

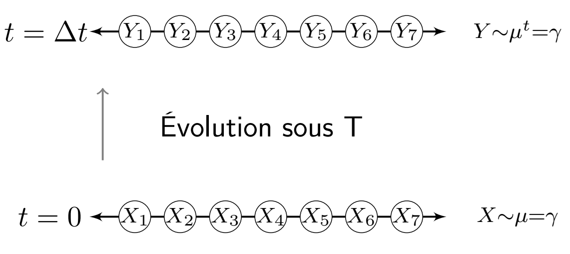
Question principale.

Nous abordons la question suivante : étant donnée une distribution μ sur $E_\kappa^{\mathbb{Z}}$, quels sont les IPS qui possèdent μ en tant que mesure invariante ?

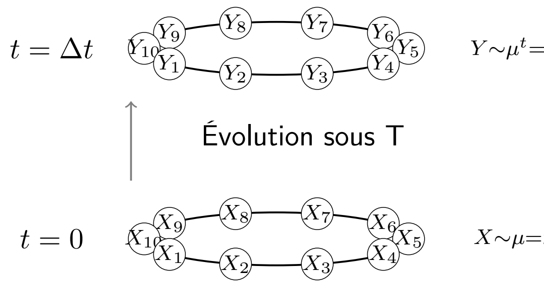
Nous répondons complètement à la question dans deux cas : nous caractérisons tous les IPS ayant μ comme distribution invariante lorsque μ est mesure produit et lorsque μ est la loi d'un processus de Markov de mémoire m sur la ligne. Nous donnons également des caractérisations similaires pour les IPS définis sur $\mathbb{Z}/n\mathbb{Z}$ (et aussi sur $\mathbb{Z}^d$).

D'habitude les résultat de calculs explicites de mesures invariantes dans la littérature sont obtenus modèle par modèle, ici notre approche permet de calculer les mesures invariantes d'un nombre infini de modèles présentant certaines caractéristiques algébriques.

Notre contribution :
Nous présentons un bilan des résultat obtenus. C'est important de noter que la même matrice de taux de saut peut être utilisée pour définir des IPS dans différents graphes, étant donné que les règles d'évolution sont locales ($L < \infty$) (voir fig. 1).



(a) Esquisse d'une portion finie du processus évoluant sur la ligne avec mesure invariante γ.



(b) Esquisse du processus évoluant sur le cycle de longueur 10 avec mesure invariante ν.

FIGURE 1 – Les IPS à droite et à gauche sont définis à partir de la même matrice T (le temps évoluant de bas en haut).

On dit qu'un processus $(X_k, k \in \mathbb{Z})$ possède une **distribution de Markov (MD)** (ρ, M) (de mémoire $m = 1$) avec noyau de Markov $M := [M_{i,j}]_{i,j \in E_\kappa}$ et distribution initiale $\rho \in \mathcal{M}(E_\kappa)$ si

$$\mathbb{P}(X[\![0, n]\!] = x) = \rho_{x_0} \prod_{j=0}^{n-1} M_{x_j, x_{j+1}}, \quad \text{pour tout } n \text{ et tout } x \in E_\kappa^{n+1}.$$

Un processus $(X_k, k \in \mathbb{Z}/n\mathbb{Z})$ à valeurs sur $E_\kappa^{\mathbb{Z}/n\mathbb{Z}}$ possède une **distribution de Gibbs** $Gibbs(M)$ sur $\mathbb{Z}/n\mathbb{Z}$ et avec noyau $M := [M_{i,j}]_{i,j \in E_\kappa}$ si

$$\mathbb{P}(X = x) = \frac{\prod_{j=0}^{n-1} M_{x_j, x_{j+1 \bmod n}}}{\text{Trace}(M^n)}, \quad \text{pour tout } x \in E_\kappa^{\mathbb{Z}/n\mathbb{Z}}.$$

Théorème : *Soient κ fini, $L = 2$ et M un noyau de Markov sur E_κ. Si M possède des entrées positives et ρ est la unique mesure invariante de M, les conditions suivantes sont équivalentes pour le pair (T, M) :*

1. *La distribution de Markov (ρ, M) est invariante par* T *sur la ligne $\mathbb{Z}$.*
2. *$Gibbs(M)$ est invariante par* T *sur $\mathbb{Z}/n\mathbb{Z}$, pour tout $n \geq 3$.*
3. *$Gibbs(M)$ est invariante par* T *sur $\mathbb{Z}/7\mathbb{Z}$.*

Comments :

1. L'importance de ce théorème vient du fait que les équations de stabilité des lois finies dimensionnelles forme un système d'équations de taille infinie et de degré non borné en M, alors que l'invariance de $Gibbs(M)$ sur le cycle de longueur 7 est explicite, fini, linéaire en T de degré de 7 en M.
2. En autres termes, nous avons relié l'invariance de MD (ρ, M) sur la ligne avec l'invariance du $Gibbs(M)$ sur le cycle. Si le processus défini par T sur le cycle de longueur 7 possède une distribution invariante de Gibbs avec un noyau positif M, alors tous les IPS définis à partir de T sur les cycles de longueur $n \geq 3$ ont aussi une distribution Gibbs avec noyau M.

3. Sous certaines conditions supplémentaires, le théorème est valable pour un nombre infini de couleurs, c'est-à-dire $\kappa = \infty$.
4. Nous donnons également une "condition nécessaire et suffisante" pour l'invariance d'une distribution de Markov M pour toute mémoire $m \geq 0$ et pour toute rang $L \geq 2$.
5. Nous donnons quelques connexions entre le théorème précédent et le matrix ansatz (voir [32]).

Autres resultats :

— Conditions nécessaires et suffisantes pour l'existence de mesures produits, invariantes sur $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$ et $\mathbb{Z}^d$.

— La caractérisation complète des mesures invariantes pour $m = 1$, $L = 2$ et $\kappa = 2$.

— Un algorithme pour trouver toutes les MD invariantes pour une matrice T donnée sur la ligne.

— Quelques applications sur différents modèles.

B. Survival and coexistence for spatial population models with forest fire epidemics. Travail en commun avec A. Linker et D. Remenik.

La modélisation d'espèces (biologiques) en concurrence pour l'espace ou les ressources est un sujet d'étude actif en biologie mathématique. Les modèles classiques ne permettent pas d'expliquer la biodiversité, car dans les modèles introduits jusqu'à présent, l'espèce la mieux adaptée domine et conduit les autres à l'extinction. Afin de créer des modèles réalistes et de promouvoir la coexistence, des extensions ont été explorées, telles que l'ajout de prédateurs [92, 66, 101], de fluctuations aléatoires sur l'environnement [115, 86] et de maladies [67, 98]. D'autres travaux incluent l'ajout de l'effet de surpeuplement, qui prend en compte le fait que les fortes densités de population augmentent la proximité (promiscuité) des individus, ce qui facilite la propagation des maladies (voir, par exemple, [63, 102, 53]).

Nous étendons les travaux de Durrett & Remenik de [42] qui ont étudié le comportement d'un processus de particules inspiré par les spongieuses (type de papillon de nuit), dont la croissance naturelle de la population conduit à la formation d'amas géants qui sont anéantis par les épidémies (cet effet est modélisé par un processus appelé feux de forêt dans la littérature [38]). Nous étendons ce modèle dans deux directions : à plusieurs espèces en concurrence pour l'espace dans un environnement commun, et nous généralisons les taux des épidémies qui non seulement attaqueront les amas géants, mais aussi les amas de plus petit ordre.

Le modèle Multi Moth (MMM) : Soit $G_N = (V_N, E_N)$ un graphe fixé avec N sommets et soit $m \geq 1$ le nombre d'espèces dans le modèle. Le MMM est un processus de Markov à temps discret $(\eta_k)_{k\geq 0}$, où

$$\eta_k = (\eta_k(x) : x \in V_n),$$

prenant des valeurs dans $\{0, \ldots, m\}^{V_N}$, où $\eta_k(x) = i$ si, au moment k, x est occupé par un individu de type i si $i \in \{1, ..., m\}$ ou vacant si $i = 0$. Le processus $(\eta_k : k \geq 0)$ est défini à l'aide de 3 familles de paramètres :

$$\vec{\beta} = (\beta_1, \ldots, \beta_m) \in \mathbb{R}_+^m \quad , \quad \vec{\alpha}_N = (\alpha_N^1, \ldots, \alpha_N^m) \in [0,1]^m \quad \text{et} \quad \{\mathcal{N}_N(x) \subset V_N : x \in V_N\},$$

où ce dernier paramètre $\mathcal{N}_N(x)$ est un voisinage (dit de croissance) de x.

Étant donnée une configuration initiale $\eta_0 \in \{0, \ldots, m\}^{V_N}$, la dynamique du processus à chaque unité de temps est divisée en deux étapes consécutives :

Croissance : Chaque individu de type i présent sur un site $x \in V_N$ meurt, mais avant cela, il envoie un nombre aléatoire de descendants de moyenne $\beta_i > 0$ (indépendamment des autres sites) à des sites choisis de manière uniforme et aléatoire dans $\mathcal{N}_N(x)$ (ces actions sont effectuées simultanément pour tout type et tout site). Après cela, le type d'un site est choisi uniformément au hasard parmi le type des individus qu'il a reçu ; s'il n'en a reçu aucun, le type est 0.

Épidémie : Chaque site x occupé par un individu de type i *après l'étape de croissance* est attaqué par une épidémie avec probabilité α_N^i indépendamment des autres sites. Un individu infecté dans x meurt avec l'ensemble de la composante connexe de sites occupés par des individus du même type. Cela se produit indépendamment pour $i = 1, \dots, m$ et pour tout x.

Nos résultats principaux sont :

- Cas mono espèce, $m = 1$: Nous obtenons le diagramme complet de survie et extinction selon les valeurs des paramètres du modèle :
  - (Extinction) Lorsque $\beta(1-\alpha) < 1$, le temps moyen d'extinction est sous logarithmique en terme du nombre de sommets.
  - (Survie) Si $\beta(1-\alpha) > 1$, le temps moyen d'extinction est au moins une fonction linéaire en terme du nombre de sommets.
- Cas deux espèces, $m = 2$: Nous obtenons des régions (explicites) pour la coexistence ou pour la domination d'une espèce sur l'autre (avec disparition de l'autre espèce) :
  - (Domination) Pour certain paramètres, le temps moyen d'extinction du type 1 est au plus d'ordre N alors que le type 2 survit pendant au moins une période de temps égal à $e^{\theta_{\underline{\alpha}}(N)}$ pour une fonction θ explicite.
  - (Coexistence) Pour certain paramètres, avec une probabilité élevée les deux espèces sont présentes dans le système pendant une période de temps d'ordre au moins égal à $e^{\theta_{\underline{\alpha}}(N)}$ pour une fonction θ explicite.

C. Cartes planaires decorées par arbres : combinatoire.

Travail en commun avec A. Sepúlveda.

Une carte planaire enracinée est une paire $(\mathrm{m}, \vec{e})$ composée de : une carte m, qui est le plongement d'un graphe planaire fini connecté dans le plan (ou la sphère), sans croisements d'arêtes, et une arête orientée $\vec{e}$ de m (l'arête racine). Ces objets sont considérés à homéomorphisme préservant l'orientation et $\vec{e}$ près (c'est-à-dire en respectant l'ordre cyclique d'arêtes autour chaque sommet). Une carte (ignorez les couleurs pour le moment) est affichée dans Figure 2, où son arête racine est représentée par une flèche.

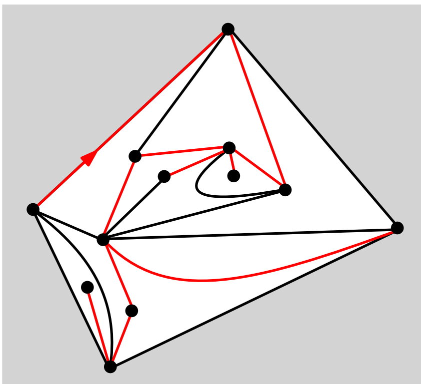


FIGURE 2 – Une carte décorée par un arbre couvrant.

Le degré d'une face est le nombre d'arêtes qui lui sont adjacentes (une arête incluse dans une face est comptée deux fois). Une q-angulation est une carte dont les faces ont degré q (Figure 2 montre une 4-angulation ; elles sont aussi appelées quadrangulations).

Un arbre enraciné, est une carte enracinée à une face. Le nombre d'arbres à n arêtes, est donné par le n-ième nombre Catalan (le graphe rouge dans Figure 2).

La face qui se trouve à gauche de l'arête racine est appelée face-racine (face en gris dans Figure 2). Dans ce qui suit, les cartes à bord sont des cartes où la face racine joue un rôle particulier ; l'ensemble des arêtes qui lui sont adjacentes forment le bord.

Toutes les autres faces sont appelées faces internes. Par exemple, une quadrangulation à bord de taille p est une carte où toutes les faces internes ont degré 4 et la face-racine a degré p.

Le bord d'une carte est dit simple s'il forme une courbe simple dans le plan. Une carte m_1 est dite une sous-carte de m_2, si m_1 peut être obtenu à partir de m_2 en supprimant des arêtes et des sommets (si la sous-carte contient la racine, elle est enracinée, sinon elle n'est pas enracinée).

Notre contribution

La principale contribution de notre travail est de présenter une nouvelle famille d'objets, les cartes décorées par un arbre, et de donner une bijection à partir de laquelle on peut les étudier.

Définition : *Pour* $(f, a) \in (\mathbb{N}^*)^2$, *une carte* (f, a) *décorée par un arbre est une paire* $(\mathsf{m}, \mathfrak{t})$ *où* m *est une carte enracinée avec* f *faces, et* $\mathfrak{t}$ *est un arbre enraciné avec* a *arêtes, de sorte que* $\mathfrak{t}$ *est une sous-carte de* m*, et que l'arête racine de la carte et l'arête racine de l'arbre coïncident.*

Une caractéristique importante des modèles de quadrangulations (f, a) décorées par un arbre uniforme est qu'elles interpolent, lorsque a varie de 1 à $f+1$, entre les quadrangulations uniformes avec f faces et les quadrangulations décorées par un arbre couvrant uniformes avec f faces.

Notre principal résultat est une bijection, à partir de laquelle nous obtenons de nombreuses formules de dénombrements de cartes décorées, des résultats combinatoires (de décomposition) et de l'information pour étudier grandes cartes aléatoires décorées par un arbre uniforme comme nous le verrons dans le résumé du travail suivant.

Proposition : *Il existe une bijection explicite* g *pour tout* $(a, f) \in (\mathbb{N}^*)^2$ *entre : l'ensemble des cartes* (f, a) *décorées par un arbre et le produit cartésien entre l'ensemble des cartes enracinées à bord simple de taille* $2a$ *et* f *faces intérieures, et l'ensemble des arbres enracinés avec* a *arêtes.*

D. Cartes planaires aléatoires décorées par arbres : limites locales.

Travail en commun avec A. Sepúlveda.

Le but de ce travail est de décrire la limite locale des différents modèles présentés dans la section précédente. Le principal résultat est que les triangulations et les quadrangulations décorées par un arbre à f faces, bord de longueur p et décorées par un arbre à a arêtes convergent dans la topologie locale vers différentes limites, selon le comportement fini ou infini de f, p et a.

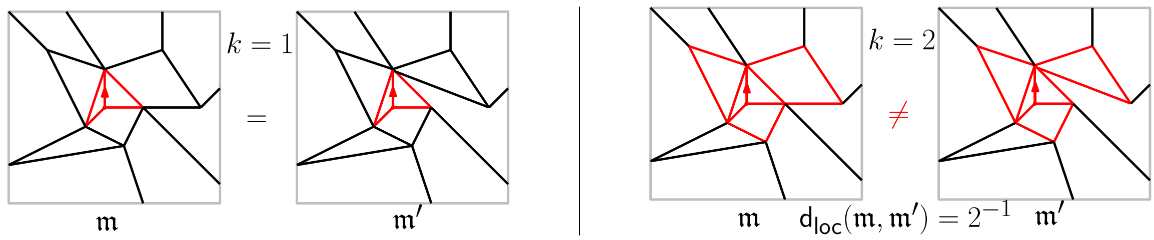
Topologie locale : Pour une carte enracinée m et $r \in \mathbb{N}$, $[\mathsf{m}]_r$ désigne la carte obtenue en considérant toutes les faces de m dont les sommets sont tous à distance inférieure à r du sommet racine de m.

Soit $\mathcal{M}$ une famille de cartes enracinées localement finies. La topologie locale sur $\mathcal{M}$ est la topologie induite par

$$\mathsf{d_{loc}}(\mathsf{m}_1, \mathsf{m}_2) = (1 + \sup\{\mathsf{r} \geq 0 : [\mathsf{m}_1]_\mathsf{r} = [\mathsf{m}_2]_\mathsf{r}\})^{-1}$$

Une séquence de cartes $(\mathsf{m}_i)_{i \in \mathbb{N}}$ converge pour $\mathsf{d_{loc}}$ ssi pour tout $r \in \mathbb{N}$, $[\mathsf{m}_i]_r$ est constant à partir d'un certain temps.

Nous définissons pour $p \in \mathbb{N}^*$ et $f \in \mathbb{N}^*$ l'ensemble $\mathcal{B}_{f,p}^{\mathcal{T},a}(q)$ de toutes les q-angulations à bord simple $(\mathsf{m}^b, \tau_N, \vec{e})$, où $(\mathsf{m}^b, \vec{e})$ est une carte enracine à bord simple de taille p ayant f faces et arête-racine $\vec{e}$ appartenant au bord, et τ_N est un arbre à a arêtes qui intersecte le bord seulement au sommet racine de m^b. Nous désignons par $\mathcal{B}_{f,0}^{\mathcal{T},0}a(q) := \mathsf{B}_f^{\mathsf{T},a}(q)$ l'ensemble des q-angulations (f, a)



Calcul de la limite locale entre $\mathfrak{m}$ et $\mathfrak{m}'$. En rouge $[\mathfrak{m}]_r$ et $[\mathfrak{m}']_r$.

décorées par un arbre. Finalement nous désignons par

$$T^{\mathsf{T},a}_{f,p} = \text{ unif. r,v. in } \mathcal{B}^{\mathcal{T},p}_{f,p}a(3) \text{ (triangulations)} \tag{1}$$

Notre contribution

Notre principal résultat est le suivant :

Proposition : *Pour $p_n \to p \in \mathbb{N}^+ \cup \{\infty\}$ et $a_n \to a \in \mathbb{N}^+ \cup \{\infty\}$, nous avons*

$$T^{\mathsf{T},a_n}_{f,p_n} \xrightarrow[local,f\to\infty]{(d)} T^{\mathsf{T},a_n}_{\infty,p_n} \xrightarrow[local,n\to\infty]{(d)} ap$$

Tous ces objets limites sont des triangulations infinies à bord simple décorées par un arbre. Le résultat analogue est également valable pour les quadrangulations.

À Daniela et l'avenir ...

Remerciements

Je tiens tout d'abord à remercier Jean-François Marckert (a.k.a. le chef), merci d'avoir guidé mes recherches et de m'avoir conseillé durant ces trois années, je me suis bien amusé et j'ai beaucoup appris. J'apprécie beaucoup ta disponibilité et ton engagement.

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Table des matières

Introduction

In this thesis we explore some models belonging to two different branches of probability theory: interacting particle systems and random planar maps. The results presented here include some work in progress and three articles: two devoted to the study of interacting particles systems and one dedicated to (random) planar maps. Even though we use combinatorial techniques and probability theory to study them, they are not linked.

We list a brief description of each chapter and the work in progress included

A <u>Invariant measures of discrete interacting particle systems: algebraic aspects.</u>
We obtain some necessary and sufficient conditions for some continuous time particle systems with discrete local state space, to have a simple invariant measure. By simple we mean Markov, Gibbs or product measure depending on the subjacent graph $\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$ or $\mathbb{Z}^2$. We present multiple applications of our results.

B <u>Survival and coexistence for spatial population models with forest fire epidemics.</u>
We investigate the effect on survival and coexistence of introducing forest fire epidemics to a certain two-species spatial competition model. Our main results show that, for the two-type model, there are explicit parameter regions where either one species dominates or there is coexistence; contrary to the same model without forest fires, for which the fittest species always dominates. We also characterize the survival and extinction regimes for the particle system with a single species. In both cases we prove the convergence of the process giving the successive proportions of individuals of each species to the orbits of a dynamical system.

C <u>Tree-decorated planar maps: combinatorial results.</u>
We introduce a new model: the (f, a) tree-decorated maps, which are maps with f faces decorated by a tree with a edges. This model, when restricted to quadrangulations, interpolates between quadrangulations with f faces, when $a = 1$, and the spanning tree-decorated quadrangulations, when $a = f + 1$ (also called tree-rooted quadrangulations). We obtain a bijection from which we get new combinatorial results concerning various models of decorated maps.

D <u>Tree-decorated planar maps: local limits. (work in progress).</u>
We prove that uniform tree-decorated triangulations and quadrangulations with f faces, boundary of length p and decorated by a tree of size a converge in the local topology to different limits, depending on the finite or infinite behavior of f, p and a. Several new families of objects are introduced.

Presentation of the results

This document is composed of an introduction and 4 chapters, each devoted to one research paper (written in collaboration with some co-authors). We chose to present them integrally, adding some special comments as follows:

Additional note.

> Some explanations (in dark blue) are added in order to guide readers that are not familiar with the area of each specific research. Sometimes we also add them to stress some facts, rephrasing in a detailed way some already present explanations.

Key idea.

> We include them (in dark green) in order to discuss the proofs: main ideas, difficulties, scope, etc. Sometimes they contain a sub-paragraph, that we call **Tools**, where we list some keywords to sum up the techniques and tools used to get the result.

Contributions

In order to explain the results in this introduction, we have chosen to present each article separately and with references to some statements that will be properly stated and proved in the chapters devoted to them.

A Invariant measures of discrete interacting particle systems: algebraic aspects.

Joint work with J.F. Marckert.

Introduction

In this work [50] we are interested in a special class of interacting particle systems (IPS): these processes are time continuous Markov processes $(\eta_t)_{t\in\mathbb{R}^+}$ taking their values in the set of colorings of a graph $G = (V, E)$, i.e.

$$\eta_t = (\eta_t(u), u \in V) \in E_\kappa^V,$$

where the set of colors is $E_\kappa := \{0, 1, \dots, \kappa - 1\}$ for some $\kappa \in \{\infty, 2, 3, \dots\}$. The set E_κ will be also called sometimes "local state space".

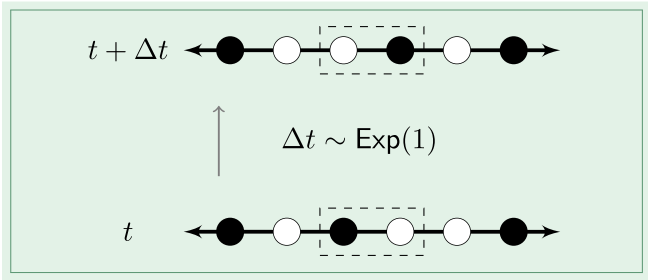
The graphs in consideration here will be the lattice $\mathbb{Z}^d$ for $d \geq 1$, $\mathbb{Z}/n\mathbb{Z}$ with $n \geq 1$ and the segment $[\![0, n]\!]$, for some $n \geq 1$.

In the case of $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ the dynamics of the process is defined by a jump rate matrix

$$\mathsf{T} = [\mathsf{T}_{[u|v]}]_{u,v\in E_\kappa^L},$$

where $L \geq 2$ is called the range of the interaction (size of the influential neighborhood). The entry $\mathsf{T}_{[u|v]}$ is the rate at which a subword u of η of length L is transformed into v. More precisely (neglecting some potential issues of global well definiteness), it is useful to imagine that each sub-configuration u with length L, of the global configuration, will be transformed into any subconfiguration v after a random time with exponential distribution of parameter $\mathsf{T}_{[u|v]}$. Here are some examples.

Example 1 (TASEP) Consider the line $\mathbb{Z}$ seen as a graph $G = (\mathbb{Z}, E)$ with set of edges $E = \{\{x, x+1\}, x \in \mathbb{Z}\}$. In this particle system each particle tries to jump to the right at rate 1 and the jump becomes effective only if the arrival site is vacant. Black (white) sites represents the presence (absence) of a particle on it. The jump rate matrix is given by



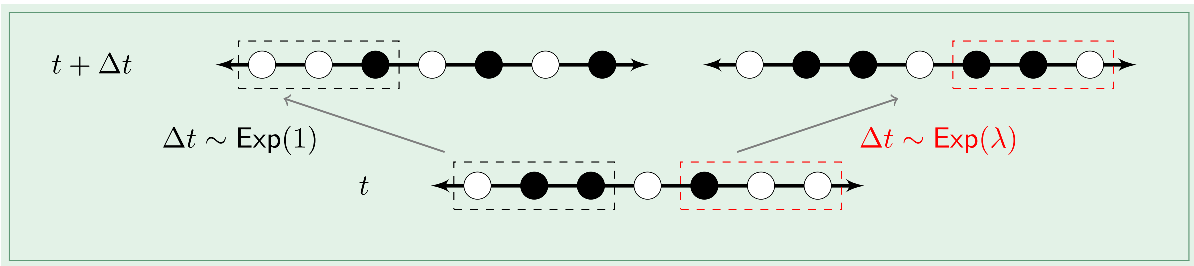
$$\mathsf{T}_{[w|w']} = \begin{cases} 1 & \text{if } w = (\bullet, \circ) \text{ and } w' = (\circ, \bullet) \\ 0 & \text{otherwise} \end{cases}$$

Meaning that each pair of sites which colors are $(\bullet, \circ)$ waits a random time exponentially distributed with rate 1 to jump to $(\circ, \bullet)$ (the colors of other sites remain the same), the exponential clocks are restarted.

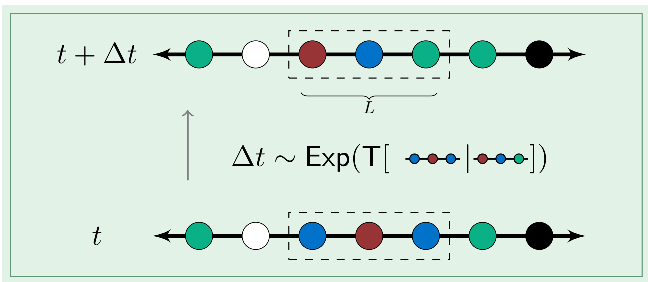
Example 2 (Contact process) Consider $G = (\mathbb{Z}, E)$ with $E = \{\{x, x+1\}, x \in \mathbb{Z}\}$. Each healthy (white) particle becomes infected (black) with rate proportional to the number of infected neighbors. Each infected particle recovers with rate 1. The jump rate matrix is given by:

$$\mathsf{T}_{[w|w']} = \begin{cases} 1 & \text{if } w = (a, \bullet, b), w' = (a, \circ, b), \text{ for any } a, b \in \{\bullet, \circ\} \\ \lambda(\mathbb{1}_{a=\bullet} + \mathbb{1}_{b=\bullet}) & \text{if } w = (a, \circ, b), w' = (a, \bullet, b), \text{ for any } a, b \in \{\bullet, \circ\} \\ 0 & \text{otherwise} \end{cases}$$

The following figure is a sketch of two possible jumps from the same configuration.



It can be noticed at this stage that different jump rate matrices, with different range L, can induce the same dynamics.



All IPS defined on $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ with finite local state spaces (finite set of colors) are well defined as Markov processes, i.e. there is a correspondence between the jump rate operator and a Markov semigroup defining a Feller process (see [84]), but this is not clear when $\kappa = \infty$ due to potential infinitely many jumps at one site in a finite time period (see for the general picture [84, 108] and [82, 4] for a specific case).

Even if the process is not defined as a Feller process (i.e. the Markovian semigroup is not in correspondence with a Markovian generator) we can make sense of a generator as follows. Define "local map"

$$m_{i,w,w'} : \begin{array}{ccc} E_\kappa^{\mathbb{Z}} & \longrightarrow & E_\kappa^{\mathbb{Z}} \\ \eta & \longmapsto & m_{i,w,w'}(\eta) \end{array}, \tag{2}$$

where $[\![a,b]\!] := [a,b] \cap \mathbb{Z}$, $\forall a < b \in \mathbb{Z}$. In words, $m_{i,w,w'}$ encodes the replacement of the subword $\eta[\![i+1, i+L]\!]$ by w', when it is equal to w. Formally, $m_{i,w,w'}$:
– keeps η unchanged (i.e. $m_{i,w,w'}(\eta) = \eta$), if the subword $\eta[\![i+1, i+L]\!] \neq w$
– changes $\eta[\![i+1, i+L]\!]$ to w', if the subword $\eta[\![i+1, i+L]\!] = w$.

Define the generator

$$(Gf)(\eta) = \sum_{(i,w,w') \in \mathbb{Z} \times E_\kappa^2} \mathsf{T}_{[w|w']} \left[f(m_{i,w,w'}(\eta)) - f(\eta) \right], \tag{3}$$

acting on continuous functions f from the set of configurations to $\mathbb{R}$, encoding somehow the complete dynamics of the IPS (see section I.1.1 for a detailed discussion)

Definition A.1
A distribution μ on E_κ^V is said to be *invariant* by the IPS with jump rate matrix T (or simply by T) if $\eta_t \overset{(d)}{=} \eta_0 \overset{(d)}{=} \mu$ for any $t \geq 0$, where $\overset{(d)}{=}$ denote the equality in distribution.

A well defined IPS is a well defined continuous Markov process, with Markovian semigroup $\{P_t\}_t \geq 0$, obtained from the jump rate operator G. We denote by $\mu^t = \mu P_t$ the distribution of the process at time t, when starting from μ. When the process is well defined, the generator G and the distributions $(\mu^t, t \geq 0)$ are linked and satisfy $\forall n_1 < n_2 \in \mathbb{Z}, \forall x[\![n_1, n_2]\!] \in E_\kappa^{[\![n_1,n_2]\!]}$:

$$\frac{\partial}{\partial t} \mu^t(x[\![n_1, n_2]\!]) = \int G \mathbf{1}_{\{w[\![n_1,n_2]\!] = x[\![n_1,n_2]\!]\}} d\mu^t(w). \tag{4}$$

This equation (known in the literature as Kolmogorov equation) represents the "fluctuations" of the finite dimensional distributions of the process (η_t) as the time passes by. For an Invariant measures μ the left hand side of eq. (4) equals zero; this together with eq. (4) give a characterization of invariant measure in terms of G, i.e. $\forall n_1 < n_2 \in \mathbb{Z}, \forall x[\![n_1, n_2]\!] \in E_\kappa^{[\![n_1,n_2]\!]}$:

$$\int G \mathbf{1}_{\{w[\![n_1,n_2]\!] = x[\![n_1,n_2]\!]\}} d\mu(w) = 0 \tag{5}$$

In our setting (product σ-algebra) if one wants to discover invariant measures, it is enough to work at the level of finite dimensional distributions, since they characterize the whole measure. Equation (5) gives some conditions on the finite dimensional distribution of invariant measures.

Bibliographical notes: In the late 1960's Frank Spitzer started to explored what he called "more elaborated random walk models" ([105]). In 1970 [106] he finally introduced several families of, what we now call, interacting particles systems. Around the same period Ronald Dobrushin published several articles (see, for example, [36, 37]) related to these, now widely studied, Markov processes. They were both finally recognized as co-founders of the theory of interacting particle systems. Some reference of this theory, including some general models, are the books of Liggett [84], Durrett [40], Kipnis & Landim [71] and Swartz [108].

As said before, IPS when we consider countably many local state space ($\kappa = \infty$) are not always well defined. Different techniques are used to define/construct IPS in different general space states setups, we comment on two of them. First, embedding the IPS on a Poisson point process, which was inspired by Harris [62]. Second, by means coming from measure theory and functional analysis (Hille-Yosida theorem). See e.g. [83, 108, 71, 4], where proofs of existence and construction can be found in some particular cases. The infinite alphabet case ($\kappa = \infty$) is treated for example in [83, Chap. IX], [71, 7, 4, 45].

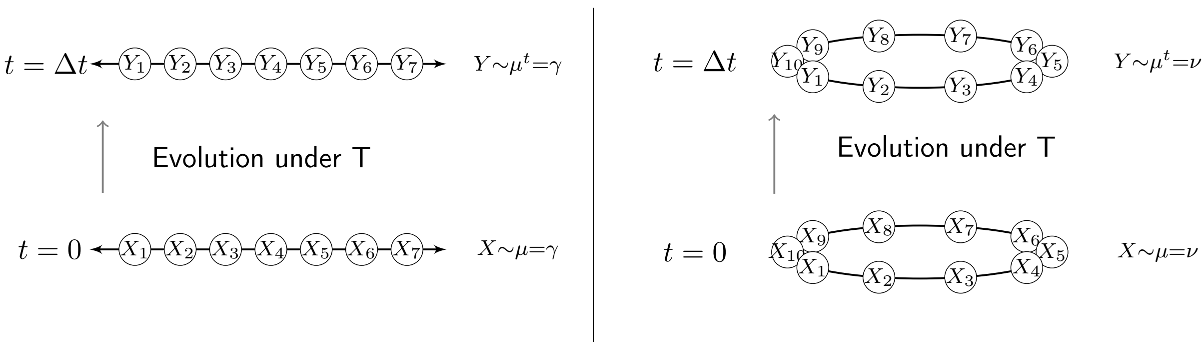
Some related works around the computation of invariant distribution(s) of a given PS, or the characterization of its uniqueness and convergence can be found in [15, 29, 45, 55]. Other questions about IPS concern the study of IPS out of equilibrium, i.e. out of the invariant measure regime, for example, their hitting time to reach a certain state and other observables [110], their mixing time [95, 10, 74, 80], etc. All these works are not directly related to the present work.

Main question/main answer

Usually for a specific IPS the natural questions about its invariant measures are related to its existence, uniqueness, ergodicity, rate of convergence, among others. Here, we work in an unusual direction, since instead of focusing on the properties of a given IPS, we consider all of them, altogether, and address the following question: given a distribution $\mu \in \mathcal{M}(E_\kappa^{\mathbb{Z}})$ (the set of measures on $E_\kappa^{\mathbb{Z}}$), what are the IPS that possess μ as an invariant distribution ?
We answer completely to the question in two cases: we characterize all IPS having μ as invariant distribution when μ is a product measure and when μ is a Markov process with memory m on the line. We also provide similar and interrelated characterizations for IPS defined on $\mathbb{Z}/n\mathbb{Z}$ (and also on $\mathbb{Z}^d$). This type of results is reminiscent of the question of integrable systems in statistical physics, where the typical results consist in finding the subset of the parameters space of the models, in which some algebraic simplifications arise, eventually leading to closed formulas.

Our contribution

It is important to remark that the same jump rate matrix can be used to define IPS in different graphs, given that the evolution rules are local ($L < \infty$). Because of this we will explore the link between particles system on the line and on the cycle, both defined from the same jump rate matrix (see fig. 4).



(a) Finite segment sketch of the process evolving on the line with invariant measure γ.

(b) Sketch of the process evolving on the cycle of length 10 with invariant measure ν.

Figure 4 – Right and left IPS are defined from the same jump rate matrix T (time evolving from bottom to top).

Invariance of product measures

Invariant product measures $\rho^{\mathbb{Z}}$ for $L = 2$ are characterized by the following (eq. (5)): $\forall x[\![1,n]\!] \in E_\kappa^n, \forall n \in \mathbb{N}$

$$\sum_{x_0,x_{n+1}\in E_\kappa} \sum_{j=0}^{n} \left(\sum_{u,v\in E_\kappa} \mathsf{T}_{[u,v|x_j,x_{j+1}]} \Big(\prod_{\substack{0\le k\le n+1 \\ k\notin\{j,j+1\}}} \rho_{x_k} \Big) \rho_u \rho_v - \left(\prod_{k=0}^{n+1} \rho_{x_k} \right) \sum_{u,v\in E_\kappa} \mathsf{T}_{[x_j,x_{j+1}|u,v]} \right) = 0 \tag{6}$$

which is an infinite system of equations. Each equation can be read as the balance between the infinitesimal mass creation and mass destruction of the word $x[\![1,n]\!]$, where the left term inside the parenthesis (6) measures creation and the right term measures destruction. Of course, under the product measure, the mass of $x[\![1,n]\!]$ is $\rho_{x_1}\rho_{x_2}\dots\rho_{x_n}$. The range of the second sum in (6) runs over all possible position where a jump is possible and since $L = 2$, jumps in x_1 and x_n may occur from T acting in (x_0, x_1) and (x_n, x_{n+1}), respectively; for this reason we need to sum over all possible values of x_0 and x_{n+1}.

In the case of the cycle of length n, invariant product measures $\rho^{\mathbb{Z}/n\mathbb{Z}}$ for $L = 2$ are characterized by the following (coming from eq. (5)): $\forall x \in E_\kappa^{\mathbb{Z}/n\mathbb{Z}}$

$$\sum_{j=0}^{n-1} \left(\sum_{u,v\in E_\kappa} \mathsf{T}_{[u,v|x_j,x_{j+1}]} \Big(\prod_{\substack{0\le k\le n-1 \\ k\notin\{j,j+1\}}} \rho_{x_k} \Big) \rho_u \rho_v - \left(\prod_{k=0}^{n-1} \rho_{x_k} \right) \sum_{u,v\in E_\kappa} \mathsf{T}_{[x_j,x_{j+1}|u,v]} \right) = 0, \tag{5'}$$

where the index numbers are taken $\mod n$. This equation can be again read as the balance between mass creation and mass destruction of the word x. The main difference with (6) is that in (5') is no longer needed to sum over boundary values.

The first result we get is about the invariance of product measures: we prove that $\rho^{\mathbb{Z}}$ is invariant for the IPS defined on the line iff $\rho^{\mathbb{Z}/3\mathbb{Z}}$ is invariant for the IPS defined on the cycle of length 3, therefore establishing the algebraic reduction of an infinite algebraic system with unbounded degree in the ρ_i's to a finite one.

Theorem A.2: See Theorem I.2.2.2 for the complete statement
If E_κ is finite, the range L is 2, T is a jump rate matrix of range L, and ρ is a measure with support E_κ, then the following are equivalent:

1. The product measure $\rho^{\mathbb{Z}}$ is invariant by T (for the IPS defined on $\mathbb{Z}$).
2. The product measure $\rho^{\mathbb{Z}/n\mathbb{Z}}$ is invariant by T for $n \ge 2$ (for the IPS defined on $\mathbb{Z}/n\mathbb{Z}$).
3. The product measure of $\rho^{\mathbb{Z}/3\mathbb{Z}}$ is invariant by T (for the IPS defined on $\mathbb{Z}/3\mathbb{Z}$).

A classical sufficient condition is given by the so called **detailed balance equations**: the product measure $\rho^{\mathbb{Z}}$ is invariant by T on $\mathbb{Z}$ if:

$$\rho_b\rho_c \mathsf{T}_{[b,c|u,v]} = \rho_u\rho_v \mathsf{T}_{[u,v|b,c]} \text{ for any } b,c,u,v \in E_\kappa. \tag{7}$$

This is equivalent to the invariance of the product measure on an interval of length 2 by T. Theorem A.2 implies that the "detailed balance equation" is just a sufficient condition, not a necessary one. Theorem I.2.2.2 gives the complete conditions.

Invariance of Markov measures

Recall that a process $(X_k, k \in \mathbb{Z})$ is said to have a Markov distribution (MD) (ρ, M) (of memory $m = 1$) with Markov kernel $M := [M_{i,j}]_{i,j\in E_\kappa}$ and a initial distribution $\rho \in \mathcal{M}(E_\kappa)$ if

$$\mathbb{P}(X[\![0,n]\!] = x) = \rho_{x_0} \prod_{j=0}^{n-1} M_{x_j,x_{j+1}}, \qquad \text{for any } n \text{ and any } x \in E_\kappa^{n+1}.$$

A process $(X_k, k \in \mathbb{Z}/n\mathbb{Z})$ with values on $E_\kappa^{\mathbb{Z}/n\mathbb{Z}}$ is said to have a Gibbs distribution $Gibbs(M)$ on $\mathbb{Z}/n\mathbb{Z}$ with kernel $M := [M_{i,j}]_{i,j\in E_\kappa}$ if

$$\mathbb{P}(X = x) = \frac{\prod_{j=0}^{n-1} M_{x_j,x_{j+1 \bmod n}}}{\mathrm{Trace}(M^n)}, \qquad \text{for any } x \in E_\kappa^{\mathbb{Z}/n\mathbb{Z}}.$$

Consider $L = 2$ and $m = 1$ (one of the simplest non-trivial cases). When working on the line, the invariance of a Markov distribution (ρ, M) is characterized by the following system of equations (coming from eq. (5)):

$$\sum_{\substack{x_{-1},x_0,\\ x_{n+1},x_{n+2}\in E_\kappa}} \sum_{j=0}^{n} \sum_{u,v\in E_\kappa} \mathsf{T}_{[u,v|x_j,x_{j+1}]} \rho_{x_{-1}} \Big(\prod_{\substack{-1\le k\le n+1\\ k\notin\{j-1,j,j+1\}}} M_{x_k,x_{k+1}} \Big) M_{x_{j-1},u} M_{u,v} M_{v,x_{j+2}}$$
$$- \sum_{\substack{x_{-1},x_0,\\ x_{n+1},x_{n+2}\in E_\kappa}} \Big(\rho_{x_{-1}} \prod_{k=-1}^{n+1} M_{x_k,x_{k+1}} \Big) \sum_{j=0}^{n} \sum_{u,v\in E_\kappa} \mathsf{T}_{[x_j,x_{j+1}|u,v]} = 0, \qquad \forall x[\![1,n]\!] \in E_\kappa^n, \forall n \in \mathbb{N} \tag{8}$$

Again these equations can be read the balance between mass creation and mass destruction as we did in eq. (6). The boundary effects here are "worse" because of the memory of the Markov process. This infinite system of equations has unbounded degree in M and is linear in T. Solving such a system, that is finding all pairs (M, T) such that the the MD (ρ, M) is invariant by T, is not an easy task a priori: the main result of this part is to bring a complete solution to this problem.

Now we are in position to state our main theorem.

Theorem A.3: See Theorem I.2.1.2 for the complete statement
Let κ be finite, $L = 2$ and let M be a Markov kernel on E_κ. If M has positive entries and ρ is the unique invariant measure of M then the following statements are equivalent for the couple (T, M):

1. The Markov distribution (ρ, M) is invariant by T on the line $\mathbb{Z}$.
2. $Gibbs(M)$ is invariant by T on $\mathbb{Z}/n\mathbb{Z}$, for all $n \geq 3$.
3. $Gibbs(M)$ is invariant by T on $\mathbb{Z}/7\mathbb{Z}$.

Comments:

1. The importance of this theorem comes from the fact that the invariance system on the line has an infinite number of equations with unbounded degree in M, while the invariance of $Gibbs(M)$ on the cycle of length 7 is finite. More specifically, the system is explicit, finite, linear on T and has degree 7 in M.
2. In words, we linked the invariance of the MD (ρ, M) running on the line with the invariance of $Gibbs(M)$ running on the cycle. If the process defined by T on the cycle of length 7 has an invariant Gibbs distribution with a positive kernel M, then all IPS defined from T on cycles of length $n \geq 3$ also have a Gibbs distribution with the same kernel as invariant distribution.

3. Under some additional conditions, the theorem still holds for the infinite number of colors case, i.e. $\kappa = \infty$ (Section I.3.2).
4. We provide also an "iff condition" (Theorem I.3.1.2) for the invariance of a Markov distribution M for any memory $m \geq 0$ [1] and for any range $L \geq 2$.

   In this case there exists a specific number $h := 4m + 2L - 1$ replacing 7 in theorem A.3, meaning that invariance of a MD with kernel M by an IPS with jump rate matrix with range L is equivalent to the invariance of $Gibbs(M)$ in the cycle with length $4m + 2L - 1$.
5. In section I.3.5 we give some connections between Theorem A.3 and the matrix ansatz (See [32]). These connections are made by establishing relations of invariance equations (8) for different sizes from Theorem A.3, which finally relates the invariant measures for different sizes, as the matrix ansatz does.

Key idea.

> We transform the statement of Theorem A.3 into an algebraic one by expressing the balance of finite dimensional distributions. We manipulate the difference between creation-destruction equations (eq. (8)) for different (well chosen) words. These differences are shown to be related with the invariance equation of the cycle. The most difficult part of Theorem A.3 is to show that the invariance on the cycle of $Gibbs(M)$ implies the invariance of the MD (ρ, M) on the line.

In the case $m = 1$ and $L = 2$ we provide an algorithm to find the set of all Markov distributions which are invariant by a given jump rate matrix T on the line. It relies on the following theorem.

Theorem A.4: See Theorem I.2.5.1 for the complete statement
Let $\kappa < \infty$ and ν be a probability measure invariant by T on $\mathbb{Z}/3\mathbb{Z}$. If there exists a positive recurrent Markov kernel $M = (M_{a,b})_{a,b \in A}$ such that for a normalization constant t

$$\nu_{a,b,c} = t^{-3} M_{a,b} M_{b,c} M_{c,a} \text{ for any } (a,b,c) \in E_\kappa^3, \tag{9}$$

then, M is unique and has an explicit expression depending only on ν (this expression appears in Theorem I.2.5.1).

Algorithm to find the set of all Markov distributions invariant by T on the line

i) Compute the set of probability measures ν invariant on the cycle of length 3 (this is a linear problem in ν), details given in eq. (I.43).

ii) For each ν found in i) compute M solving (9), this can be done using linear algebra from the explicit expression (see theorem I.2.5.1).

iii) For each M found in ii) test if $Gibbs(M)$ is invariant by T on $\mathbb{Z}/7\mathbb{Z}$ (depending on the algebraic nature of the solution, this can be done directly or by using Gröbner basis tools).

In the literature, one does not find many cases in which explicit invariant distributions have been found: in general, the invariant measures of IPS are complex to find and to prove in absence of criteria. We give one, which makes the task easier. The general results that are known deal with product measures in some mass transport models (see [4, 45]). In other cases only some properties are known, for example, the existence of a non-trivial invariant measure for the contact process

1. The notion of Markov distribution can be generalized to memory $m \in \{0, 1, 2, \cdots\}$ if for $\mathbb{P}(X_k \in A \,|X_{k-i}, i \geq 1) = \mathbb{P}(X_k \in A \,|X_{k-i}, 1 \leq i \leq m) = M_{x[\![k,k-m+1]\!],x[\![k-1,k-m]\!]}$, for any k.

running on $\mathbb{Z}$ (see [84]), but no information about the form of this measure is known (we add some in Corollary A.6).

A set of configurations A is called absorbing if leaving A has probability 0, i.e. $\eta_0 \in A \implies \eta_t \in A,\ \forall t \geq 0.$

The following result is obtained from our main theorem Theorem A.2 and gives some immediate applications.

Theorem A.5: See Theorem I.2.1.5 for the complete statement
Let $\kappa < \infty$. Consider a jump rate matrix T with range L, which is not identically 0.
Suppose that for infinitely many integers n the IPS defined on the cycle $\mathbb{Z}/n\mathbb{Z}$ with jump rate matrix T possesses a proper absorbing set S_n subset of the configuration set $E_\kappa^{\mathbb{Z}/n\mathbb{Z}}$. Under these conditions, there does not exist any Markov distribution (with any memory m), with full support, invariant by T on the line.

Corollary A.6: See Corollary I.4.1.2 and Corollary I.4.1.3 for the complete statement
The voter model [a] and the contact process do not have a Markov distribution of any memory $m \geq 0$ as invariant distribution on the line.

a. The voter model is an IPS in which colors represents opinions: each site copies the opinion of one random neighbor at rate 1.

Other applications obtained:

1. We find parameters of some IPS having a hidden Markov chain as invariant distributions.
2. An introduction to Gröbner basis, the main tools to solve explicit algebraic equations, is given in section I.4.1. Using it, the case $\kappa = 2$ and $L = 2$ is totally explicitly solved (section I.4.3), that is the manifold on which lives the parameters T$'s$ having an invariant product measure on the line is given.

Invariance of product measures on $\mathbb{Z}^d$

Here we consider IPS indexed by $\mathbb{Z}^d$, whose configuration space is of course $E_\kappa^{\mathbb{Z}^d}$. We suppose that the jump rate matrix instead of characterizing "the jump rate of size L-subwords" is defined for configurations of the d-dimensional hypercube HC$[L,d] = [\![0, L-1]\!]^d$ for some L as follows

$$\mathsf{T} = \left((\mathsf{T}_{[w|w']})_{w,w' \in E_\kappa^{\mathsf{HC}[L,d]}} \right)$$

We can reduce our attention to the case of hypercubes, since any (more general) shape F is a subset of an hypercube H large enough. For example for $d = 2$, HC$[2,2]$ is the square Sq formed by the cells $(0,0), (1,0), (1,1)$ and $(0,1)$. An example of jump rate matrix is the following T with all entries equal to 0, except

$$\mathsf{T}\left[\begin{array}{c|c} 1\,1 & 0\,1 \\ 0\,0 & 0\,1 \end{array}\right] = 1, \mathsf{T}\left[\begin{array}{c|c} 0\,1 & 0\,0 \\ 0\,1 & 1\,1 \end{array}\right] = 1, \mathsf{T}\left[\begin{array}{c|c} 0\,0 & 1\,0 \\ 1\,1 & 1\,0 \end{array}\right] = 1, \mathsf{T}\left[\begin{array}{c|c} 1\,0 & 1\,1 \\ 1\,0 & 0\,0 \end{array}\right] = 1, \qquad (10)$$

meaning that the 2×2 square sub-configurations having exactly two ones in adjacent positions (not diagonally), jump at rate 1 to the clockwise rotation of the sub-configuration. The system of equation that characterizes a product invariant measure on $\mathbb{Z}^d$ has an infinite number of equations on the variables $\{\rho_u\}_{u \in E_\kappa}$ and $\{\mathsf{T}_{[w|w']}\}_{w,w' \in E_\kappa^{\mathsf{HC}[L,d]}}$.

Theorem A.7: See Theorem I.2.3.1 for the complete statement
When $\kappa < +\infty$, a product measure $\rho^{\mathbb{Z}^d}$ is invariant by T if and only if $\rho^{\mathbb{Z}}$ solves an explicit finite system of equations on the variables $\{\rho_u\}_{u\in E_\kappa}$ and $\{\mathsf{T}_{[w|w']}\}_{w,w'\in E_\kappa^{\mathsf{HC}[L,d]}}$.

Example of application of Theorem A.7 : For the example defined in eq. (10), all Bernoulli product measures with any parameter in $(0,1)$ are invariant. This can be obtained from Theorem A.7, or as a consequence of reversibility. For the slightly generalized jump rate matrix with zero entries except for

$$\mathsf{T}\begin{array}{|c|c|}\hline 1\,1 & 0\,1\\ 0\,0 & 0\,1\\\hline\end{array} = a,\ \mathsf{T}\begin{array}{|c|c|}\hline 0\,1 & 0\,0\\ 0\,1 & 1\,1\\\hline\end{array} = b,\ \mathsf{T}\begin{array}{|c|c|}\hline 0\,0 & 1\,0\\ 1\,1 & 1\,0\\\hline\end{array} = c,\ \mathsf{T}\begin{array}{|c|c|}\hline 1\,0 & 1\,1\\ 1\,0 & 0\,0\\\hline\end{array} = d, \tag{11}$$

two situations are possible: using Theorem A.7

1. $a = b = c = d$, then all Bernoulli product measures with any parameter $\rho \in (0,1)$ are invariant.
2. Otherwise, there is no invariant product measure.

Reversibility cannot be used to prove the second fact.

B Survival and coexistence for spatial population models with forest fire epidemics.

Joint work with A. Linker and D. Remenik.

Introduction

One active subject of study in mathematical biology is the modeling of species competing for space or resources. The classical models are inadequate to explain biodiversity, since in the so far introduced models the fittest species dominates and drives the others to extinction. In order to make realistic models and promote coexistence, some extensions have been explored as the addition of predators [92, 66, 101], of random fluctuations in the environment [115, 86] and of diseases [67, 98]. Other works include the addition of the crowding effect, which takes into account that high population densities increase the connectedness of individuals, which makes easier to spread diseases (see, for example, [63, 102, 53]).

We extend the work of Durrett & Remenik in 2009 [42] where they studied the behavior of a finite particle system inspired by gypsy moths, whose population's natural growth leads to the formation of giant clusters which are wiped out by epidemics, decreasing the population [2]. The model we propose extends this model in two directions: multiple species competing for the space in a common environment and generalization of the epidemics which will not only attack giant clusters, but also clusters of smaller size.

As a side remark, we want to add that forest fire models, which were first introduced in [38], have received much interest as a prime example of a system showing self-organized criticality, see e.g. [97].

This research is motivated by the question of existence of "simple models" for which biodiversity occurs. The multi-type extension we propose, where the addition of forest fires in this discrete-time multi-type contact process indeed generates (long time) coexistence regimes.

2. This effect is called forest fires in the literature [38].

A related work is [24], where Chan & Durrett explored a continuous time multi-type contact process in $\mathbb{Z}^2$ with the addition of forest fires, which kill individuals regardless of their type. Our model is different, since we work in a random environment and since the forest fires we consider kill neighbors of the same type.

The Multi Moth Model (MMM): Fix a graph $G_N = (V_N, E_N)$ with N vertices and let $m \geq 1$ be the number of species in the model. The MMM is a discrete time Markov process $(\eta_k)_{k \geq 0}$, where

$$\eta_k = (\eta_k(x) : x \in V_n),$$

taking values in $\{0, \ldots, m\}^{V_N}$, where $\eta_k(x) = i$ if, at time k, x is occupied by an individual of type i if $i \in \{1, ..., m\}$ or vacant if $i = 0$. The process $(\eta_k : k \geq 0)$ is defined using 3 families of parameters:

$$\vec{\beta} = (\beta_1, \ldots, \beta_m) \in \mathbb{R}_+^m \quad , \quad \vec{\alpha}_N = (\alpha_N^1, \ldots, \alpha_N^m) \in [0,1]^m \quad \text{and} \quad \{\mathcal{N}_N(x) : x \in V_N\},$$

where this last parameter $\mathcal{N}_N(x) \subset V_N$ is a set containing x.

Consider an initial configuration $\eta_0 \in \{0, \ldots, m\}^{V_N}$.

The dynamics of the process at each time step is divided into two consecutive stages:

Growth: Each individual of any type i present at any site $x \in V_N$ dies, but before that it sends a $Poisson(\beta(i))$ number of descendants (independently from the others sites), to sites chosen uniformly at random in $\mathcal{N}_N(x)$, the *growth neighborhood of* x (these actions are performed simultaneously).

After that, the individual that survives is chosen uniformly at random among the individuals it received (this fixes the type); if none, the type is 0.

Epidemic: Each site x occupied by an individual of type i *after the growth stage* is attacked by an epidemic with probability α_N^i, independently across sites. An infected individual at x then dies along with its entire connected component of sites occupied by individuals of type i. This happens independently for $i = 1, \ldots, m$ and for all x.

Notice that in the single type model ($m = 1$), $\vec{\alpha}_N$ and $\vec{\beta}$ are no longer vectors, we write α_N and β instead.

In the previous work [42] Durrett and Remenik showed that in the case $m = 1$ and for a randomly chosen 3-regular connected graph on N vertices G_N, under some growth conditions on $\mathcal{N}_N$ and α_N, when $\alpha_N \to 0$ and $\mathcal{N}_N(x) = G_N$ (mean field regime), the sequence of densities[3] $(\rho_k^N, k \geq 0)$ defined by (recall, only type 1)

$$\rho_k^N := \frac{1}{N} \sum_{x \in V_N} \mathbb{1}_{\eta_k^N(x)=1}, \tag{12}$$

(where k denotes the time), converges as $N \to \infty$ to the orbit of a deterministic dynamical system which, for certain parameters (α, β), is chaotic. It is worth noting that as α_N goes to zero, small order clusters are less likely to be attacked and in the limit basically only "infinite clusters" are successfully wiped-out.

3. this is the proportion of occupied sites.

Our results

We keep considering as in [42] G_N as a uniform 3-regular connected graphs on N vertices.

Convergence:

Define $\rho_k^{N,(i)}$ as the density of type i at time k similarly to eq. (12). We extend the convergence to a dynamical system as follows

Theorem B.1
Consider the MMM with $m \geq 1$ types and with $\mathcal{N}_N(x) = B(x, r_N)$. Suppose that the sequences $\vec{\alpha_N}$ and r_N satisfy

$$\alpha_N(i) \xrightarrow[N\to\infty]{} \alpha(i) \in [0,1], \quad \alpha_N(i) r_N \xrightarrow[N\to\infty]{} \infty, \quad \text{and} \quad \sqrt{\frac{r_N}{\alpha_N(i)}} \leq \frac{\log_2(N)}{5} \quad \forall N \in \mathbb{N}. \tag{13}$$

Suppose also that η_0^N is a given by a product measure where each site is independently chosen to have type i with probability p_i. Then as $N \to \infty$, the density process $\left(\vec{\rho}_k^N\right)_{k\geq 0}$ associated to the MMM converges in distribution (on compact time intervals) to the deterministic orbit, starting at $\vec{p} = (p_1, \dots, p_m)$, of the dynamical system DS(h).

Notation: $\vec{\rho}_k$ is a stochastic process and its time dependence is written, as usual, as sub-index, while $\vec{p}^{\,k}$ is a dynamical system value, where time is the k-th composition of a function written, as usual, as super-index.

Key idea.

> We propose a candidate limiting dynamical system and then we prove that, in fact, it is the limit. The technical assumption
>
> $$\sqrt{\frac{r_N}{\alpha_N(i)}} \leq \frac{\log_2(N)}{5} \quad \forall N \in \mathbb{N}$$
>
> helps to create a close modified version of the process at each point ignoring the epidemics from the outside of its ball of radius $\sqrt{r_N/\alpha_N(i)}$. Since 3-random regular graphs look like a 3-regular tree in a neighborhood of radius $\frac{\log_2(N)}{5}$, we can guess the limit candidate from ordinary generating functions of 3-regular trees.
> **Tools:** Ordinary generating function, percolation on vertices.

The condition $\vec{\alpha}_N \to \vec{\alpha} \in (0,1)^m$ implies that in the limit the epidemic attacks not only giant clusters, but of small order too.

It is important to remark that even though, the diameter of a uniform random 3-regular graph is $O(\log(N))$ [16], the condition on r_N is not trivial (i.e. it is not of the order of the diameter):

— If $\alpha(i) \in (0,1]$ for all $i \in \{1,2,\dots,m\}$, the conditions on r_N simplifies and we can consider any $(r_N)_{N\in\mathbb{N}}$ going slowly enough to infinity. Ex: $r_N = \log(\log(N))$.

— If $\alpha(i) = 0$, then the MMM with parameters $\alpha_N(i) = \log(\log(\log(N)))^{-1}$ and $r_N = \log(\log(N))$ satisfies the hypothesis of the theorem.

One species ($m = 1$): survival and extinction regime:

Recall that in this case we denote $\vec{\alpha}_N = \alpha_N$ and $\vec{\beta} = \beta$. Define the extinction time

$$\tau_N = \inf\{k \geq 0 : \rho_k^N = 0\}. \tag{14}$$

The following theorem states that there is a dichotomy, based on the value

$$\phi(\alpha_N, \beta) := \beta(1 - \alpha_N),$$

between extinction ($\phi(\alpha_N, \beta) \leq 1$) and survival ($\phi(\alpha_N, \beta) > 1$).

Theorem B.2: See Theorem II.2.2.2 for the complete statement

In the mean field regime (i.e. $\mathcal{N}_N(x) = V_N$) MM and any $N \in \mathbb{N}$ we have:

(i) (Extinction) If $\phi(\alpha_N, \beta) \leq 1$, then for all $n \in \mathbb{N}$ and any initial density ρ_0^N

$$\mathbb{P}(\tau_N \geq n) \leq \begin{cases} 1 - (1 - \phi(\alpha_N, \beta)^n)^N & \text{if } \phi(\alpha_N, \beta) < 1, \\ 1 - \left(1 - \frac{2}{n(1-\alpha_N)(\sigma^2 + \alpha_N \beta^2)}\right)^N & \text{if } \phi(\alpha_N, \beta) = 1, \end{cases}$$

where σ^2 is the variance of the offspring distribution of each particle in the growth stage. In particular, it follows that when $\phi(\alpha_N, \beta) < 1$ there is $C > 0$ independent of N such that

$$\mathbb{E}(\tau_N) \leq C \log(N). \tag{15}$$

(ii) (Survival) If $\phi(\alpha_N, \beta) > 1$ and $\rho_0^N \geq \bar{\rho}_0$ for some $\bar{\rho}_0 > 0$, then there exists $c > 0$ (depending only on $\bar{\rho}_0$ and α_N) such that

$$\mathbb{P}(\tau_N \geq n) \geq \left(1 - \frac{c}{N}\right)^{3n}.$$

In particular, if we assume that $\alpha_N \log_2(N) \to \infty$ then

$$\mathbb{E}(\tau_N) \geq \frac{N}{4c}. \tag{16}$$

Key idea.

> In the extinction regime, we dominate the process by individuals surviving in a Galton-Watson process which gives the bounds on the absortion. For the survival regime, it is enough to keep track of isolated occupied sites.
> **Tools:** Galton-Watson process, Tchebychev inequality, Chernoff bounds, coupling, independent graph set.

We believe that in the survival setting it holds that the mean extinction time is at least exponential. Even though we could not prove this in the general setting, we showed that for an explicit $b_1 \in (0, 1)$ if $b_1\phi(\alpha_N, \beta) > 1$, then the assertion holds (Theorem II.2.2.3). It is worth remarking that this is obtained without using the behavior of the limiting dynamical system.

The bifurcation diagram of a dynamical system depending on one parameter is the plot of the orbits of the system in the fiber of each parameter value in an interval (the abscissa). We remark that for $m = 1$ and for each fixed $\alpha \in (0, 1)$ the bifurcation diagram of this dynamical system, with respect to β, as β grows, seems to develop bifurcation cascades (also known as *period-doubling bifurcations*) as shown in fig. 5. This result is not proved in the paper due to some technical obstructions.

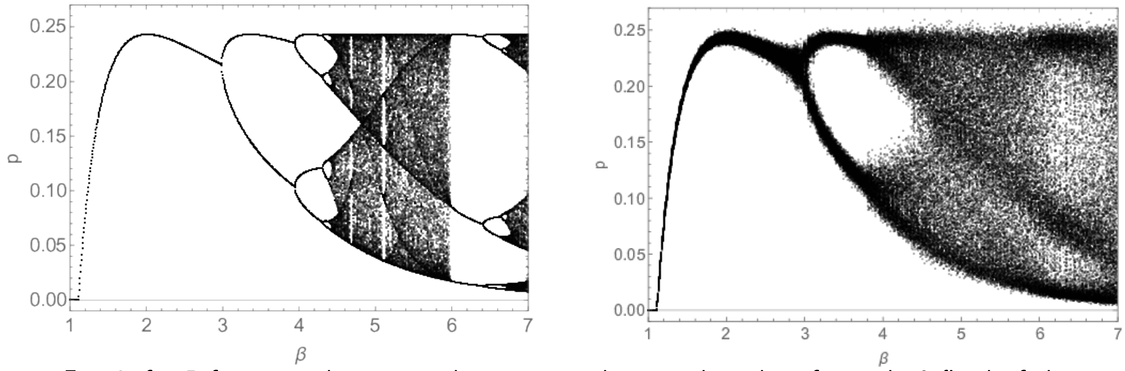


Figure 5 – Left: Bifurcation diagram with $\alpha = 0.1$, showing the orbits for each β fixed of the system between iterations 900 and 1000 (transient regime). Right: Bifurcation diagram associated to the orbits of the densities in the stochastic process for $\alpha = 0.1$ and different values of β, from iteration 900 to 1000. Here N takes values in $\{20000, 40000, 100000\}$, depending on β.

Multiple species $m = 2$: domination and coexistence.

In the MMM if one suppresses the epidemic stage, then our process is a variant of a multi-type contact process for which it can be proved that the fitter species (i.e. the one with the largest growth parameter β_i) will outcompete and drive to extinction all the other ones [94]. In the main result (multi species case) we show that the introduction of forest fire dynamics changes this picture, allowing coexistence even when the species have different fitnesses.

From our convergence theorem, to have a clue of how the process behaves, it suffices to explore the properties of the limiting dynamical system orbits.

Note that for $m = 2$, the orbits of the dynamical system at time is a vector $\vec{p}^k = (p_1^k, p_2^k)$

Theorem B.3: See Theorem II.2.3.1 for the complete statement
For the two species model ($m = 2$), a non-trivial region (determined by curves) in the space of parameters $(\alpha_i, \beta_i)_{i\in\{1,2\}}$ and $p_1^0 > 0$, there exists a real number $\bar{p}^1 > 0$ depending only on p_1^0 such that $p_1^k > \bar{p}^1$ for all k.

Theorem B.4: See Theorem II.2.3.3 for the complete statement
For the two species model and a a non-trivial region (determined by curves) in the space of parameters $(\alpha_i, \beta_i)_{i\in\{1,2\}}$ and for all initial density $\vec{p}^0$ with $p_2^0 \in (0, 1)$ we have

$$p_1^k \xrightarrow[k\to\infty]{} 0 \quad \text{and} \quad \liminf_{k\to\infty} p_2^k > 0.$$

Our main result states that these properties are inherited by the sequence of densities of the process running on G_N in a large time scale (the convergence result Theorem ?? implies the convergence for any fixed time window). More formally, let

$$\theta_\alpha(N) = \begin{cases} e^{-\sqrt{\log(N)}} & \text{if } \alpha = 0 \\ N^{-\alpha/5} & \text{if } \alpha > 0, \end{cases}$$

and $\tau_N^{(i)}$ to be the extinction time associated to type i as in eq. (14).

Theorem B.5: See Theorem II.2.3.5 for the complete statement
For the two species model and mean-field regime, assume that the proportion of individuals of each type converges, i.e. $\vec{\rho}_0^N \to \vec{p}_0$ [and suppose property II.8 is satisfied], then there exist $C > 0$, $\gamma \in (0,1)$ and $N \in \mathbb{N}$ such that for $\underline{\alpha} = \min\{\alpha_1, \alpha_2\}$ we have:

1. If the conditions of Theorems B.4 or B.3 are satisfied, then for all $n > N$
$$\mathbb{P}(\tau_N^2 \geq n) \;\geq\; \big(1 - C\theta_{\underline{\alpha}}(N)\big)^n. \tag{17}$$
2. If the conditions of Theorem B.3 are satisfied, then for all $n > N$
$$\mathbb{P}(\tau_N^1 \geq n) \;\geq\; \big(1 - C\theta_{\underline{\alpha}}(N)\big)^n, \tag{18}$$
3. If the conditions of Theorem B.4 are satisfied, then for all $n > N$
$$\mathbb{P}(\tau_N^1 \geq n) \;\leq\; N\gamma^n + \big(C\theta_{\underline{\alpha}}(N)\big)^n. \tag{19}$$

In other words we obtain that

1. Under the conditions of Theorem II.2.3.1 there is coexistence, in the sense that with high probability both species are present in the system for an amount of time of order at least $e^{\theta_{\underline{\alpha}}(N)}$.
2. Under the conditions of Theorem II.2.3.3 there is domination, in the sense that, with high probability, the extinction time of type 1 is at most of order N while type 2 survives for at least an amount of time of order at least $e^{\theta_{\underline{\alpha}}(N)}$.
3. From the theorem we deduce that there exist some parameters (α^1, β_1),$(\alpha^{1'}, \beta_1')$ and (α^2, β_2), satisfying $\phi(\alpha^2, \beta_2) > 2\log(2)$ and $\phi(\alpha^1, \beta_1) < \phi(\alpha^{1'}, \beta_1') < \phi^2(\alpha^2, \beta_2)$, such that:
   - In the MMM associated to $\vec{\alpha} = (\alpha^1, \alpha^2)$ and $\vec{\beta} = (\beta_1, \beta_2)$, type 2 dominates over type 1.
   - The MMM associated to $\vec{\alpha} = (\alpha^{1'}, \alpha^2)$ and $\vec{\beta} = (\beta_1', \beta_2)$ is in the coexistence regime.

   Meaning that tuning the values of α^1 and β_1, such that the value of $\phi(\alpha^1, \beta_1)$ increases without going over $\phi(\alpha^2, \beta_2)$ can create coexistence. This can be achieved, moreover, when $\alpha^1 = \alpha^2 = 0$.

Key idea.

> We prove in Theorem II.2.1.3 that with high probability, when the size of the graph is big enough, the process and the dynamical system are close at each step. We use this to infer the domination and coexistence regime of the dynamical system.
> **Tools:** Dynamical systems, basin of attraction, attraction/repulsion of fixed points, Galton-Watson process.

C Tree-decorated planar maps: combinatorial results.

Joint work with A. Sepulveda

This section is part of a project where we investigate the limit of random tree-decorated maps. In this chapter we state a bijection that let us extract properties of tree-decorated maps and allows the counting of different families.

The work that we present in the next section uses these combinatorial studies as a tool to help for the description of local limits, in distribution, of several families of decorated maps, with size going to ∞.

Introduction

A rooted planar map is a pair $(\mathfrak{m}, \vec{e})$ made of: a map $\mathfrak{m}$, which is an embedding of a finite connected planar graph in the plane (or the sphere), without edge crossings, and an oriented edge $\vec{e}$ of $\mathfrak{m}$ (the root-edge), considered up to direct homeomorphisms of the sphere preserving the oriented edge $\vec{e}$ too. A map (omit the colors for the moment) is shown in Figure 6, where its root edge is represented by an arrow.

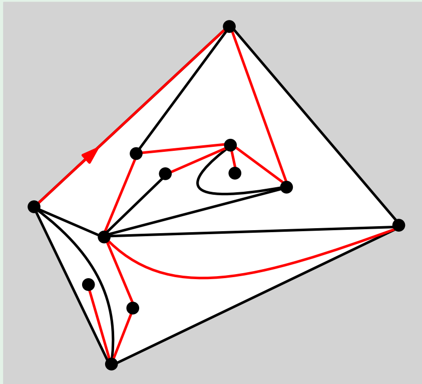


Figure 6 – A map decorated by a spanning tree.

The degree of a face is the number of edges adjacent to it (an edge included in a face is counted twice). A q-angulation is a map whose faces have degree q (Figure 6 shows a 4-angulation; these are also called quadrangulations).

A rooted plane tree, or tree for short, is a rooted map with one face (the red graph in Figure 6). The number of trees with n edges, is given by the nth Catalan number.

Unless otherwise stated, in the following all maps and all trees are rooted.

The face that is at the left of the root-edge will be called the root-face (face in gray in Figure 6). In what follows maps with a boundary are maps where the root face plays a special role; the set of edges that are adjacent to it forms the boundary. The boundary will be seen sometimes as a path around the root face, up to cyclic rotation of the indices, or as a set of edges.

All others faces are called internal faces. For example, a quadrangulation with a boundary of size p is a map where all internal faces have degree 4 and the root-face has degree p.

The boundary of a map is said to be simple if it forms a non vertex-intersecting path. A map $\mathfrak{m}_1$ is said to be a submap of $\mathfrak{m}_2$, if $\mathfrak{m}_1$ can be obtained from $\mathfrak{m}_2$ by suppressing edges and vertices (if the submap contains the root it is rooted, otherwise it is unrooted).

Definition C.1
For $(f, a) \in (\mathbb{N}^*)^2$, an (f, a) tree-decorated map is a pair $(\mathfrak{m}, \mathfrak{t})$ where $\mathfrak{m}$ is a rooted map with f faces, and $\mathfrak{t}$ is a rooted tree with a edges, so that $\mathfrak{t}$ is a submap of $\mathfrak{m}$, and so that the root-edge of the map and the root-edge of the tree coincide.

Beware that the tree is not necessarily a spanning tree.

Each time that we speak about scaling limit, we mean that we are working with the Gromov-Hausdorff topology, which is a topology on the space of isometry classes of compact metric spaces. In this case, maps are seen as metric spaces with set of points given by the vertex set and with distance given by the graph distance.

The study of random maps scaling limits started with the case of uniform quadrangulations. They are mainly studied by using the Cori-Vauquelin-Schaeffer bijection [100], which is a bijection connecting well-labeled trees and quadrangulations. Their scaling limit, the Brownian map, is a continuous metric space that was defined by Marckert and Mokkadem [88] and its convergence was proven later:

Theorem C.2: Le Gall [77] and Miermont [89]
Let q_f denote a uniform rooted quadrangulation with f faces, then

$$\left(q_f, \frac{\mathrm{d}_{q_f}}{(8/9)f^{1/4}}\right) \xrightarrow[GH]{(d)} \text{Brownian map}, \quad \text{as } f \to \infty$$

See fig. 7 for a simulation of the limit. In the past decade, the Brownian map has been intensively studied: it is homeomorphic to the two dimensional sphere [78, 90], its Hausdorff dimension is 4 [76], it can be endowed with a conformal structure linked with Liouville quantum gravity [91].

The scaling limit of $q_{f,p(f)}$, a uniform rooted quadrangulation with f internal faces and with a general boundary of size $2p(f)$, was studied by Bettinelli [13], when $f \to \infty$, where he obtained three different behaviors depending on $p(f)$.

Theorem C.3: Bettinelli [13], Bettinelli & Miermont [14]
For a sequence $(p(f))_{f\in\mathbb{N}}$, define $\bar{p} = \lim p(f) f^{-1/2}$ as $f \to \infty$, then we have

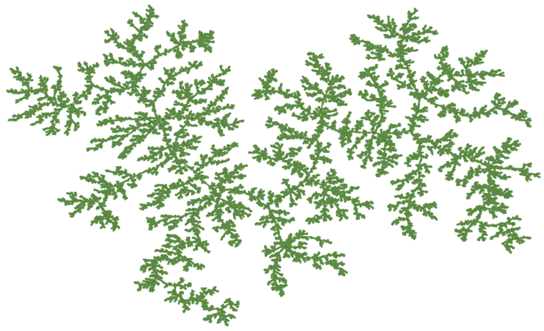
$$\left(q_{f,p(f)}, \frac{\mathrm{d}_{map}}{s(f,p(f))}\right) \xrightarrow[GH]{(d)} \begin{cases} \text{Brownian map} & \text{if } \bar{p} = 0 \text{ where } s(f,p(f)) = (8/9)f^{1/4} \\ \text{Brownian disk} & \text{if } \bar{p} \in (0,+\infty) \text{ where } s(f,p(f)) = (8/9)f^{1/4} \\ \text{CRT} & \text{if } \bar{p} = \infty \text{ where } s(f,p(f)) = 2p(f)^{1/2} \end{cases},$$

where the CRT is Aldous continuous random tree, a random continuous metric space without cycles and connected [3]. See fig. 7 for some simulations of some of these limits.

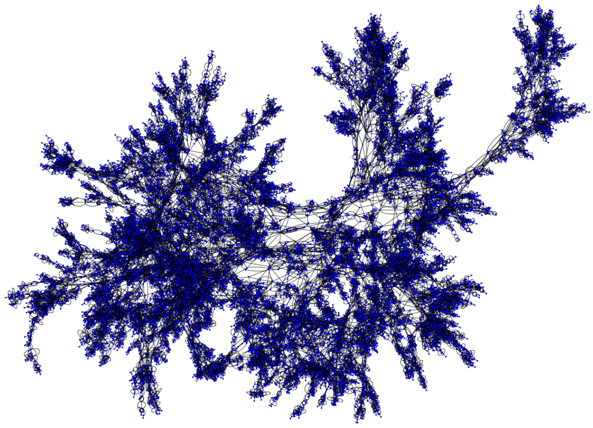
The Brownian disk is a.s. homeomorphic to the closed unit disk in $\mathbb{R}^2$, its boundary is a.s. simple, its interior has Hausdorff dimension 4 and its boundary has Hausdorff dimension 2 [13], it is related with the Brownian map, in particular the complement of a given ball in the Brownian map is a union of Brownian disks [79].

From a physical and mathematical perspective it is interesting to equip discrete maps with measures different from the uniform distribution. In order to do that, it is standard to weight them with probability distributions proportional to the partition function of some models coming from statistical mechanics.

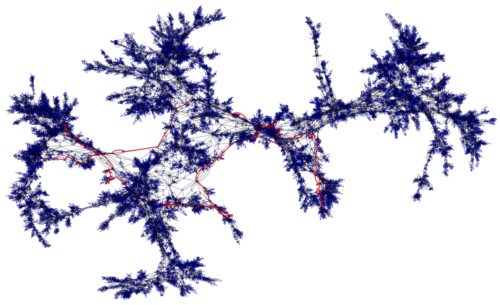
One model is the uniform ST-decorated maps which are maps chosen uniformly among all the maps with e edges decorated by a spanning tree. Uniform ST-decorated maps are studied because they are important in Euclidean 2D statistical physics. The conjectured scaling limit of ST-decorated maps (see fig. 7 for a simulation of this limit) is related to continuum Liouville quantum gravity (see [34]). Not many metric properties are known about this object, however, recently it has been shown that there exists a constant $0.275 \leq \chi \leq 0.288$, such that their expected diameter is of order f^{χ} [34, 59]. For the asymptotics ST-decorated map as a metric space, there is evidence that the limit is not the Brownian map (for example, the Brownian map and the spanning-tree decorated map have different limits in the peano-sphere topology [60]).



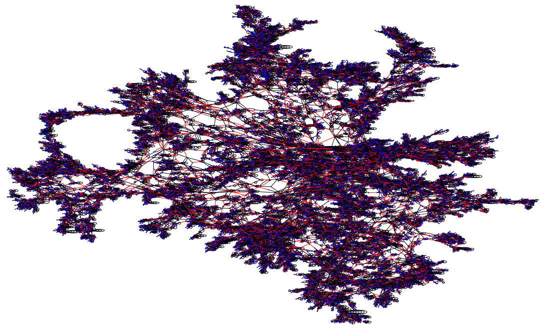
Uniform tree 50k edges.



Uniform quadrangulation with 30k faces.



Uniform quadrangulation with boundary: 30k faces- 173 boundary edges in red.



Spanning tree-decorated quadrangulation: 100k edges. Tree in red.

Figure 7 – Sketch of the possible limiting metric spaces

Notice that from the Euler's characteristic formula, when all the faces have the same fixed degree, conditioning on the number of total edges and the total number of faces is the same.

An important characteristic of the uniform (f, a) tree-decorated quadrangulation model is that it interpolates, when a varies from 1 to $f+1$, between uniform quadrangulation with f faces and the uniform ST-decorated quadrangulation with f faces. We expect, in the scaling limit, that this model interpolates between the Brownian map and the scaling limit of the uniform ST-decorated map. In light of this effect, we hope to give a phase transition between these objects obtaining an insight about the scaling limit of uniform ST-decorated maps.

Our contribution

Bijection and counting results.

The first result is a bijection linking the tree-decorated maps with maps with a simple boundary and trees.

Proposition C.4: See Theorem III.1.2.2 for the complete statement

There exists an explicit bijection g for any $(a, f) \in (\mathbb{N}^*)^2$ between: the set of (f, a) tree-decorated maps and the Cartesian product between the set of rooted maps with a simple boundary of size $2a$ and f interior faces and the set of rooted trees with a edges.

Key idea.

One direction of the bijection consists in gluing the simple boundary and the tree following the contour of the tree. The other direction consist in keeping the tree and "ungluing" the contour of the tree to create a boundary.
Tools: Tree contour, equivalence relation.

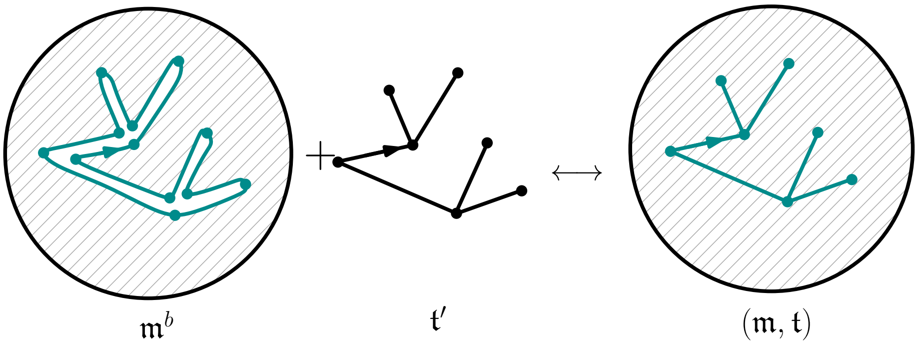


Figure 8 – Sketch of the bijection. Left: rooted map with simple boundary and planar rooted tree drawn in a suggestive way. Right: Tree decorated map. We plot it being embedded in the sphere. The arrows are root-edges and the grid lines represent the inner faces. From left to right we call it the gluing procedure and ungluing procedure, in the other sense.

Some comments about the bijection:

For a decorated map, we call internal vertices and internal edges to those that do not intersect the decoration. In the case of a map with a boundary we call internal the vertices and edges that do not intersect the boundary and we call internal the faces that are different from the root-face.

The gluing function g makes a correspondence between:

Tree-decorated map		**[Map with a boundary, Tree]**
Faces of degree q	$\longleftrightarrow$	Internal faces of degree q
Internal vertices of degree d	$\longleftrightarrow$	Internal vertices of degree d
Internal edges	$\longleftrightarrow$	Internal edges
Corner of the tree	$\longleftrightarrow$	Boundary vertices.

Some extensions:

— The function g can be extended to consider attributes between the object that are in correspondence, for example: coloring of the faces in the tree-decorated map.

— In Theorem III.1.2.2 one can fix the number of edges instead of the number of faces: the same "gluing" function g induces (by restriction) a bijection between tree-decorated maps with e edges and tree of size a and the Cartesian product of rooted trees of size a and maps with a simple boundary of size $2a$ and $e + 2a$ total number of edges.

— We can restrict the bijection to q-angulations.

— We can restrict the bijection to decorations taken in some specific family of planar trees, for example:

- Binary tree- decorated maps.
- SAW decorated maps (Already done by Caraceni & Curien).

From this bijection one can obtain many counting formulas. We present, as an example, the following corollaries, obtained from Theorem II.1.1.2 and [12].

Corollary C.5: See Theorem III.1.2.4 for the complete statement
The number of (f, a) tree-decorated quadrangulations, where the root of the map coincides with the root of the tree, is

$$3^{f-a}\frac{(2f+a-1)!}{(f+2a)!(f-a+1)!}\frac{(3a)!}{a!(2a-1)!}\frac{1}{a+1}\binom{2a}{a}. \tag{20}$$

In the case of quadrangulations, the condition of having f faces implies that it has $f+2$ vertices and $2f$ edges.

Corollary C.6
The number of spanning tree-decorated quadrangulations with f faces, where the root of the map coincides with the root of the tree, is

$$\frac{2}{(f+1)(f+2)}\binom{3f}{f,f,f}. \tag{21}$$

The last result motivates a possible generalization of the Catalan numbers: for $n, m \in \mathbb{N}$, $m \geq 1$ define:

$$C_{m,n} = m!\left(\prod_{i=1}^{m}\frac{1}{n+i}\right)\binom{(m+1)n}{\underbrace{n,n,\dots,n}_{m+1 \text{ times}}} = \binom{m+n}{n}^{-1}\binom{(m+1)n}{\underbrace{n,n,\dots,n}_{m+1 \text{ times}}}.$$

When $m = 1$, we recover the Catalan numbers and, for this definition, $C_{2,f}$ counts the spanning tree-decorated quadrangulations where the root of the map is the same as that of the tree. These numbers are in fact integers for any $n, m \in \mathbb{N}$ for $m \geq 1$ (see proposition III.4.3.8).

D Tree-decorated random planar maps: local limits

With A. Sepulveda

The aim of this work is to describe the local limit of different models presented in the preceding section. The main result states that uniform tree-decorated triangulations and quadrangulations with f faces, boundary of length p and decorated by a tree of size a converge in the local topology to different limits, depending on the finite or infinite behavior of f, p and a.

Introduction

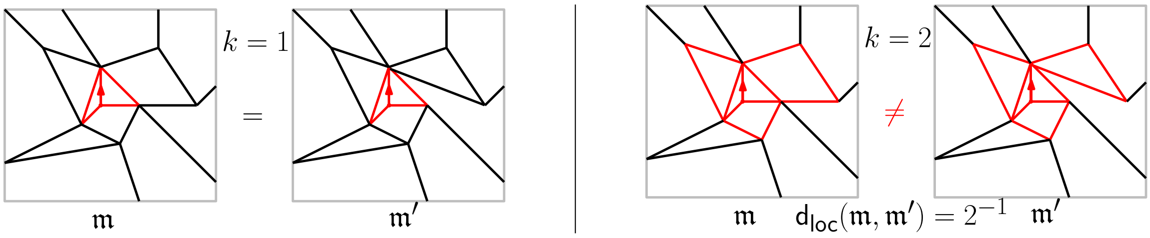
We will establish limit theorems, for the local topology.

Local topology: For a rooted map m and $r \in \mathbb{N}$, let $[\mathrm{m}]_r$ denote the map obtained by considering all the faces of m whose vertices are all at graph distance less than r from the root-vertex of m.

Consider $\mathcal{M}$ a family of locally finite rooted maps. The local topology on $\mathcal{M}$ is the topology induced by

$$\mathsf{d}_{\mathsf{loc}}(\mathsf{m}_1, \mathsf{m}_2) = (1 + \sup\{\mathsf{r} \geq 0 : [\mathsf{m}_1]_\mathsf{r} = [\mathsf{m}_2]_\mathsf{r}\})^{-1}$$

A sequence of maps $(\mathsf{m}_i)_{i\in\mathbb{N}}$ converges for $\mathsf{d}_{\mathsf{loc}}$ iff for all $r \in \mathbb{N}$, $[\mathsf{m}_i]_r$ is eventually constant. It can be proved that the space $(\overline{\mathcal{M}}, \mathsf{d}_{\mathsf{loc}})$ is Polish (metric, separable and complete).



Computation of the local distance between $\mathfrak{m}$ and $\mathfrak{m}'$. In red the $[\mathfrak{m}]_r$ and $[\mathfrak{m}']_r$.

Define for $p \in \mathbb{N}^*$ and $f \in \mathbb{N}^*$ the set $\mathcal{B}^{\mathcal{T},a}_{f,p}(q)$ of all tree-decorated q-angulations with a simple boundary $(\mathsf{m}^b, \tau_N, \vec{e})$, where $(\mathsf{m}^b, \vec{e})$ is a rooted map with a simple boundary of size p having f faces with the root-edge $\vec{e}$ in the boundary and τ_N is a tree with a edges which intersects the boundary only at the root-vertex of m^b. Denote by $\mathcal{B}^{\mathcal{T},a}_{f,0}(q) := \mathsf{B}^{\mathsf{T},a}_f(q)$ the set of all (f,a) tree-decorated q-angulations. Also define $\mathcal{SB}_{f,p}(q)$ as the set of q-angulations with a simple boundary $(\mathsf{m}^b, \vec{e})$ where m^b is a map with f faces and a simple boundary of size p. Set the following random variables

$$Q^{\mathsf{T},a}_{f,p} = \text{unif. r,v. in } \mathcal{B}^{\mathcal{T},a}_{f,p}(4) \text{ (quadrangulations)}, \tag{22}$$
$$Q_{f,p} = \text{unif. r,v. in } \mathcal{SB}_{f,p}(4) \text{ (quadrangulations)}, \tag{23}$$
$$\tau_a = \text{unif. r,v. in } \mathsf{T}_a \text{ (trees)},^{4} \tag{24}$$
$$T^{\mathsf{T},a}_{f,p} = \text{unif. r,v. in } \mathcal{B}^{\mathcal{T},a}_{f,p}(3) \text{ (triangulations)} \tag{25}$$
$$T_{f,p} = \text{unif. r,v. in } \mathcal{SB}_{f,p}(3) \text{ (triangulations)}. \tag{26}$$

One of the first results concerning the local topology is due to Kesten [70] and says that uniform rooted planar trees of size n converges in distribution for the local topology

$$\tau_n \xrightarrow[local]{(d)} \tau_\infty \tag{27}$$

where τ_∞ is the **critical geometric Galton-Watson tree conditioned to survive**. The limiting object τ_∞ is an infinite random plane tree with a.s. one infinite branch called the spine (i.e. it is one-ended).

In the setting of random triangulations with a simple boundary, Angel [5] obtained that

$$T_{f,p} \xrightarrow[local(f\to\infty)]{(d)} T_{\infty,p} \xrightarrow[local(p\to\infty)]{(d)} \mathcal{H}(3)_\infty, \tag{28}$$

where $\mathcal{H}(3)_\infty$ is the **Uniform infinite half plane triangulation with simple boundary** (also denoted as the UIHPT$_\mathsf{S}$). Both $T_{\infty,p}$ and $\mathcal{H}(3)_\infty$ are one-ended. For $p = 1$ the limiting object $T_{\infty,2}$ is called **Uniform infinite plane triangulation** UIPT.

4. The set of rooted planar trees with a edges.

In the setting of random quadrangulations with a simple boundary, Curien & Miermont [30] proved that

$$Q_{f,p} \xrightarrow[local(f\to\infty)]{(d)} Q_{\infty,p} \xrightarrow[local(p\to\infty)]{(d)} \mathcal{H}(4)_\infty, \tag{29}$$

where $\mathcal{H}(4)_\infty$ is the **Uniform infinite half plane quadrangulation with simple boundary** (also denoted as the UIHPQ$_S$). As in the case of triangulations, $Q_{\infty,p}$ and $\mathcal{H}(4)_\infty$ are one-ended. For $p = 2$ the first convergence is due to Krikun [72] and the limiting object $Q_{\infty,1}$ is the well known **Uniform infinite plane quadrangulation** UIPQ.

Our results

We prove that uniform tree decorated triangulations (quadrangulations) with a simple boundary converge in distribution with respect to the local topology to an infinite tree-decorated triangulation (quadrangulation) with a simple boundary, where different limiting objects appear depending on the behavior of the tree's size sequence (a_n) and the boundary size sequence (p_n).

Proposition D.1: See Propositions IV.2.0.1 and IV.2.0.2 for the complete statement
For $p_n \to p \in \mathbb{N}^+ \cup \{\infty\}$ and $a_n \to a \in \mathbb{N}^+ \cup \{\infty\}$, we have

$$T^{\mathsf{T},a_n}_{f,p_n} \xrightarrow[local, f\to\infty]{(d)} T^{\mathsf{T},a_n}_{\infty,p_n} \xrightarrow[local, n\to\infty]{(d)} T^{\mathsf{T},a}_{\infty,p}$$

All these limit objects are tree-decorated infinite triangulations with a simple boundary. The analog result holds for quadrangulationstoo.

As a consequence we obtain that if $a = \infty$ and $p = \infty$, the limiting object is constructed as the gluing of two independent UIHPT$_S$'s following the contour to the right and to the left of the spine of τ_∞ (see fig. 10 for an sketch of this gluing).

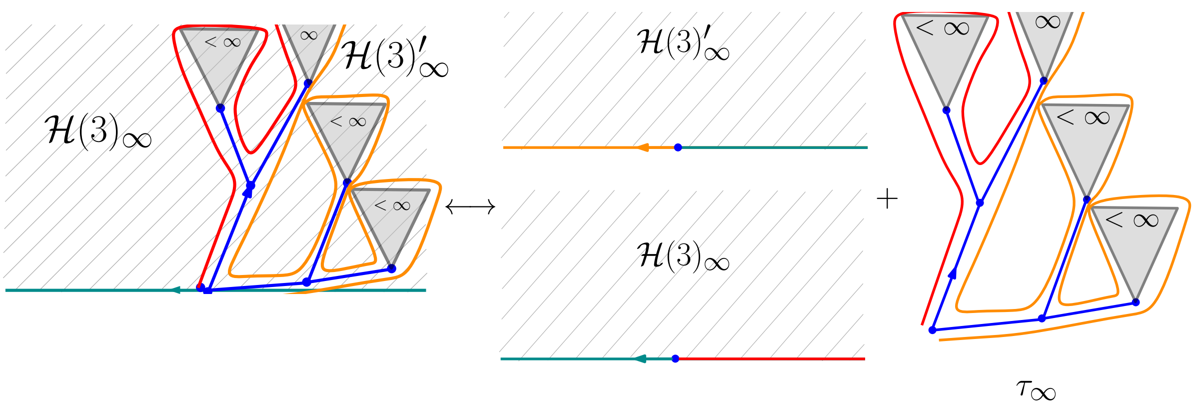


Figure 10 – Sketch of the limiting object when $a = \infty$ and $p = \infty$, constructed from two independent UIHPT$_S$'s ($\mathcal{H}(3)_\infty$ and $\mathcal{H}(3)'_\infty$) and a τ_∞. The orange and red contour lines in τ_∞ are the left and right contour around the spine. They are glued with the orange and red lines in the boundary of $\mathcal{H}(3)_\infty$ and $\mathcal{H}(3)'_\infty$.

Generalization

The proof of Proposition D.1 takes into account the one-ended behavior of τ_∞ (recall it from 27). If we want to work with distributions whose support is a set of more than one-ended trees, a generalization is needed.

For example fix $m \in \mathbb{N}^*$ consider $\tau(m)_a$ as the tree obtained by identifying the roots of m copies $t_1, \ldots, t_m$ of τ_a (uniform random tree with a edges) in such a way that around the new root, t_1 is the first tree, and after that, t_j follows t_{j-1}. A consequence of Equation (27) is

$$\tau(m)_a \xrightarrow[local]{(d)} \tau(m)_\infty, \quad \text{as} \quad a \to \infty,$$

where $\tau(m)_\infty$ is a tree, a.s. m-ended, obtained as the gluing of m (ordered) independent copies of τ_∞ glued in the root-vertex.

We present a way to generalize Proposition D.1 which applies to the case of $Q_{\infty,p+m\cdot 2a}$ a uniform infinite quadrangulations with simple boundary of size $p + m \cdot 2a$ glued with $\tau(m)_a$. We obtain that there exists a limit in distribution for the local topology as $p \to \infty$ and $a \to \infty$. We present a sketch of the limit in fig. 11.

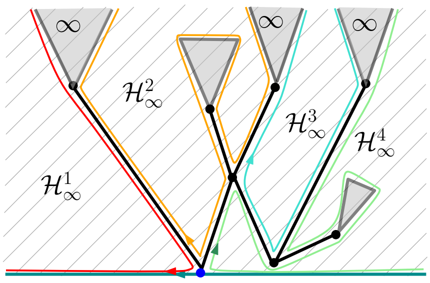


Figure 11 – Representation of the limit when gluing $Q_{\infty,p+6a}$ with $\tau(3)_a$. Here $(\mathcal{H}^i_\infty)_{i=1}^4$ are independent copies of UIHPQ$_S$ when working with quadrangulations and of the UIHPT when working with triangulations, the tree is an sketch of $\tau(3)_\infty$.

I

Invariant measures of discrete interacting particle systems: Algebraic aspects.

Contents

Consider a continuous-time particle system $\eta^t = (\eta^t(k), k \in \mathbb{L})$, indexed by a lattice $\mathbb{L}$ which will be either $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, a segment $\{1, \cdots, n\}$, or $\mathbb{Z}^d$, and taking its values in the set $E_\kappa^{\mathbb{L}}$ where $E_\kappa = \{0, \cdots, \kappa - 1\}$ for some fixed $\kappa \in \{\infty, 2, 3, \cdots\}$. Assume that the Markovian evolution of the particle system (PS) is driven by some translation invariant local dynamics with bounded range, encoded by a jump rate matrix T. These are standard settings, satisfied by the TASEP, the voter models, the contact processes... The aim of this chapter is to provide some sufficient and/or necessary conditions on the matrix T so that this Markov process admits some simple invariant distribution, a product measure (if $\mathbb{L}$ is any of the spaces mentioned above), the law of a Markov process indexed by $\mathbb{Z}$ or $[0,n]\cap\mathbb{Z}$ (if $\mathbb{L}=\mathbb{Z}$ or $\{1,\cdots,n\}$), or a Gibbs measure if $\mathbb{L}=\mathbb{Z}/n\mathbb{Z}$.

Multiple applications follow: efficient ways to find invariant Markov laws for a given jump rate matrix or to prove that none exists. The voter models and the contact processes are shown not to possess any Markov laws as invariant distribution (for any memory m) [1], which is not known to our knowledge. We also prove that some models close to these models do. We exhibit PS admitting hidden Markov chains as invariant distribution and design many PS on $\mathbb{Z}^2$, with jump rates indexed by 2×2 squares, admitting product invariant measures.

I.1 Introduction

Some notation

We let $\mathbb{N}=\mathbb{Z}^+=\{0,1,2,\cdots\}$ and $\mathbb{N}^\star=\mathbb{N}\setminus\{0\}$. For $-\infty\leq a\leq b\leq+\infty$, define $[\![a,b]\!]:=[a,b]\cap\mathbb{Z}$ as the set of integers in $[a,b]$. We will call such a set a $\mathbb{Z}$-interval.

If J is a finite subset of $\mathbb{Z}^d$ and $x\in E_\kappa^{\mathbb{Z}^d}$, then $x(J)$ stands for the sequence $(x_i, i\in J)$ sorted according to the lexicographical order of the indices, so that, for example, if $x_{(1,3)}=a$, $x_{(7,2)}=c, x_{(7,5)}=b$, then $x(\{(1,3),(7,5),(7,2)\})=(a,c,b)$. If I is a $\mathbb{Z}$-interval, for example $I=[\![3,6]\!]$, $x(I)=(x_3,x_4,x_5,x_6)$, and we will often write $x[\![3,6]\!]$ instead.

For $y=x(I)$, a sequence indexed by a set I, and for $A\subset\mathbb{Z}$, set

$$y^A=x(I\setminus A),$$

the word obtained by suppressing the letters in position belonging to A in y. Following the same idea, we denote by $M^{\{i\}}$ the matrix M with the column and row i suppressed.
For any set E, we denote by $\mathcal{M}(E)$ the set of probability measures on E (for a topology which will be specified in the context).
A function $g:A\to\mathbb{R}$ is said to be equivalent to 0, we write $g\equiv 0$, if its image is reduced to 0.

I.1.1 Models and presentation of results

All the results presented in this chapter (apart from Theorem I.2.2.6) concern space and time homogeneous particle systems (PS), with finite range interactions defined on a lattice $\mathbb{L}$, which will be $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}^d$, or a segment $[\![1,n]\!]$. The set of colors is $E_\kappa=[\![0,\kappa-1]\!]$, where κ (the number of colors) belongs to $\{2,3,\cdots\}\cup\{+\infty\}$. An element of the set of *configurations* $E_\kappa^{\mathbb{L}}$, is a coloring of the sites of $\mathbb{L}$ by the elements of E_κ (neighboring sites may have the same color). When

1. As usual, a random process X indexed by $\mathbb{Z}$ or $\mathbb{N}$ is said to be a Markov chain with memory $m\in\{0,1,2,\cdots\}$ if for $\mathbb{P}(X_k\in A\,|X_{k-i},i\geq 1)=\mathbb{P}(X_k\in A\,|X_{k-i},1\leq i\leq m)$, for any k.

well defined, the PS will be a continuous-time Markov process $\eta := (\eta^t, t \geq 0)$, where for any t, $\eta^t = (\eta^t(k), k \in \mathbb{L}) \in E_\kappa^{\mathbb{L}}$. The set $E_\kappa^{\mathbb{L}}$ is equipped with the product σ-algebra.

The construction of the family of PS considered here is illustrated on $\mathbb{Z}$ first, but considerations for the analogues on $\mathbb{Z}/n\mathbb{Z}$, $[\![1, n]\!]$ and $\mathbb{Z}^d$ will appear progressively.

Definition I.1.1.1
We call jump rate matrix (JRM) with range $L \in \mathbb{N}^\star$, a matrix

$$\mathsf{T} = \left[\mathsf{T}_{[u|v]}\right]_{u,v \in E_\kappa^L}, \tag{I.1}$$

indexed by the size L words on the alphabet E_κ, with non negative entries and with zeroes on the diagonal.

Assume for a moment **that κ, the number of colors, is finite** and fix a JRM T with range L. With any element of the "possible jumps set"

$$J = \left\{(i, w, w'), i \in \mathbb{Z}, w \in E_\kappa^L, w' \in E_\kappa^L\right\} = \mathbb{Z} \times (E_\kappa^L)^2, \tag{I.2}$$

where:
- i encodes an abscissa in an infinite word,
- w and w' encode respectively some size L initial and final words,

associate the "local map"

$$\begin{array}{rccc} m_{i,w,w'}: & E_\kappa^{\mathbb{Z}} & \longrightarrow & E_\kappa^{\mathbb{Z}} \\ & \eta & \longmapsto & m_{i,w,w'}(\eta) \end{array}, \tag{I.3}$$

which:
– if the subword $\eta[\![i+1, i+L]\!] \neq w$ keeps η unchanged (so that $m_{i,w,w'}(\eta) = \eta$)
– if the subword $\eta[\![i+1, i+L]\!] = w$, transforms this subword into w' (formally: $m_{i,w,w'}(\eta) = \eta'$ with $\eta'_j = \eta_j$ if $j \notin [\![i+1, i+L]\!]$, and $\eta'_{i+k} = w'_k$, the kth letter of w' if $1 \leq k \leq L$).

Define the generator

$$(Gf)(\eta) = \sum_{(i,w,w') \in J} \mathsf{T}_{[w|w']} \left[f(m_{i,w,w'}(\eta)) - f(\eta)\right], \tag{I.4}$$

acting on continuous functions f, for example:
– the set of bounded cylinder functions $g : E_\kappa^{\mathbb{Z}} \to \mathbb{R}$ (see e.g. the book of Kipnis & Landim [71, Section 2]) or,
– following the book of Liggett [84] (beginning p.21) or Swart [108] (starting p.72), the class C_Δ of continuous functions $g : E_\kappa^{\mathbb{Z}} \to \mathbb{R}$ such that $\sum_x \Delta_g(x) < \infty$ for

$$\Delta_g(i) = \sup\left\{|g(\eta) - g(\xi)|, \eta, \xi \in E_\kappa^{\mathbb{Z}} \text{ and } \eta(j) = \eta(i), \forall j \neq i\right\}.$$

The sum (I.4) represents a word η indexed by $\mathbb{Z}$ whose size L subwords jump: a subword equals to w is transformed into w' with rate $\mathsf{T}_{[w|w']}$ (a jump is then possible only when $\mathsf{T}_{[w|w']} > 0$). When κ is finite, such a particle system is well defined (see references given above for all details).

Many such models have been studied in the literature, for example:
- *The contact process*, for which $\kappa = 2$, $L = 3$, and all the entries of T are 0 except $\mathsf{T}_{[a,1,b|a,0,b]} = 1$

for any $(a,b)\in\{0,1\}^2$ (recovery rate), $\mathsf{T}_{[a,0,b|a,1,b]}=\lambda(a+b)$ for some $\lambda>0$ the infection rate (the same model can be expressed using a JRM with range $L=2$ instead: $\mathsf{T}_{[1,0|1,1]}=\mathsf{T}_{[0,1|1,1]}=\lambda$, $\mathsf{T}_{[1,1|0,1]}=\mathsf{T}_{[1,0|0,0]}=1$).

• *The voter model*, for which $\kappa=2$, $L=3$, $\mathsf{T}_{[a,1-c,b|a,c,b]}=\mathbf{1}_{c=b}+\mathbf{1}_{c=a}$ for any $(a,b)\in\{0,1\}^2$, the other entries of T being 0: an individual makes its neighbors adopt its opinion after an exponential random time.

• *The stochastic Ising model*, for which $\kappa=2$, $L=3$ and JRM T with zero entries except for

$$\mathsf{T}_{[a,b,c|a,1-b,c]}=e^{-\beta(2b-1)(2a+2c-2)}\text{ for any }(a,b,c)\in\{0,1\}^3. \tag{I.5}$$

Here the state 1 represents a vertex on the line with positive magnetization, 0 a vertex with negative magnetization and β a positive parameter, which, depending on its sign, favors or penalizes configurations in which vertices magnetization are aligned.

• *The TASEP* on $\mathbb{Z}$ with $\kappa=2$, $L=2$, $\mathsf{T}_{[1,0|0,1]}=1$ and the others $\mathsf{T}_{[u|v]}$ being 0.

A distribution μ on $E_\kappa^{\mathbb{Z}}$ is said to be *invariant* by T if $\eta^t\sim\mu$ for any $t\geq 0$, when $\eta^0\sim\mu$ (where the notation $\sim$ means "distributed as"). Following the discussion given below (I.4), this property can be rephrased when κ is finite, as $\int Gf d\mu=0$ for any f bounded cylinder function f (or function of C_Δ). A simple argument ([71, Lem. 1.3. p. 23]) shows that it is also characterized by $\int Gf d\mu=0$ for any indicator function f of the type

$$f(\eta)=\mathbf{1}_{\eta[\![n_1,n_2]\!]=x[\![n_1,n_2]\!]} \tag{I.6}$$

for some fixed word $x[\![n_1,n_2]\!]$ and fixed indices $n_1\leq n_2$: this is the balance between the (infinitesimal) creation and destruction of the subword $x[\![n_1,n_2]\!]$ in the interval $[\![n_1,n_2]\!]$ under the distribution μ.

Recall that under the product σ-algebra, a measure $\mu\in\mathcal{M}(E_\kappa^{\mathbb{Z}})$ is characterized by its finite dimensional distributions.

We are interested in the following question: for what JRM T does there exist a simple invariant distribution? Here the word "simple" stands for distributions as product measures, Markov laws or Gibbs measures (depending on the underlying graph where is defined the particle system). It turns out that this question has a rich algebraic nature, and we then decided to focus on this question only. The algebra in play depends on T and on the fixed family of distributions whose invariance is under investigation.

Additional note.

> Let us precise what we mean by algebraic nature. Think, for example, of a discrete time Markov process with transition matrix P, which describe the jump structure of the Markov process as T does for the particle system. A measure π, in this setting is said to be invariant for P if $\pi P=\pi$. This equation relate P and π in an algebraic manner. As we will see, there is also an algebraic relation between T and invariant measures of the particle system.

Consider a function f as given in (I.6). The **single** jumps of the PS that may affect the value of $f(\eta)$ take place in the *dependence set* of $[\![n_1,n_2]\!]$ which is larger than $[\![n_1,n_2]\!]$:

$$D[\![n_1,n_2]\!]=[\![n_1-(L-1),n_2+L-1]\!]. \tag{I.7}$$

For any w and z in $E_\kappa^{[\![n_1,n_2]\!]}$, set the induced transition rate $\mathsf{T}_{[w|z]}$ from w to z as:

$$\mathsf{T}_{[w|z]}=\sum_{[\![a+1,a+L]\!]\subset D[\![n_1,n_2]\!]}\mathsf{T}_{[w[\![a+1,a+L]\!]|z[\![a+1,a+L]\!]]}\,\mathbf{1}_{w_j=z_j\text{ for all }j\in[\![n_1,n_2]\!]\setminus[\![a+1,a+L]\!]}, \tag{I.8}$$

that is the sum of the transition rates which makes this transition possible in **a single jump totally included** in w. For a fixed pair (w, z) the contribution of the $\mathbb{Z}$-interval $[\![a+1, a+L]\!]$ is 0 if $\mathsf{T}_{[w[\![a+1,a+L]\!]|z[\![a+1,a+L]\!]]} = 0$ (jump not allowed), or if w and z do not coincide outside $[\![a+1, a+L]\!]$. This includes the case where $n_2 - n_1$ is too small, that is $< L - 1$.

Notice that taking the same notation for the transition rate between two words as for the JRM is possible since they coincide if the lengths of w and z are both L.

We want to reformulate in a Lemma what has been said so far concerning the cases where κ is finite:

Lemma I.1.1.2
Let $\kappa < +\infty$. A probability measure $\nu \in \mathcal{M}\left(E_\kappa^{\mathbb{Z}}\right)$ is invariant under T on the line if it solves the system of equations $\mathsf{Sys}(\mathbb{Z}, \nu, \mathsf{T})$ defined by

$$\left\{\mathsf{Line}^{\mathbb{Z}}(x[\![n_1, n_2]\!], \nu) = 0, \quad \text{for any } n_1 \leq n_2, \ \text{for any } x[\![n_1, n_2]\!] \in E_\kappa^{[\![n_1, n_2]\!]}\right. , \tag{I.9}$$

where

$$\mathsf{Line}^{\mathbb{Z}}(x[\![n_1, n_2]\!], \nu) = \sum_{w,z \in E_\kappa^{D[\![n_1,n_2]\!]}} \left(\nu(w)\mathsf{T}_{[w|z]} - \nu(z)\mathsf{T}_{[z|w]}\right) \times 1_{z[\![n_1,n_2]\!]=x[\![n_1,n_2]\!]}. \tag{I.10}$$

We now define the notion of **algebraic invariance** of a probability measure with respect to a particle system. The aim of this notion is to disconnect the problem of proper definition of a particle system which brings its own technical difficulties and obstructions when $\kappa = +\infty$ (see discussion in Section I.1.2) to the resolution of the systems (I.9) which is "just" an algebraic system, which can be solved independently from other considerations.

Definition I.1.1.3
For κ finite or infinite, a probability measure $\nu \in \mathcal{M}\left(E_\kappa^{\mathbb{Z}}\right)$ is said to be algebraically invariant under T on the line (we write ν is AI by T on the line) if it solves the system of equations (I.9).

Additional note.

> We reserved the name invariant, as usual, for well defined PS. Nevertheless, AI makes sense even when a PS is not well defined, where we do not have the characterization given in eq. (I.15).

Again, in the case where $\kappa < +\infty$, standard invariance of measures and algebraic invariance are equivalent notions. When $\kappa = +\infty$, difficulties arise (see Section I.1.2) and the notion of algebraic invariance is indeed useful.

Extension on $\mathbb{Z}/n\mathbb{Z}$. The previous considerations for PS η indexed by $\mathbb{Z}$ can be extended to $\mathbb{Z}/n\mathbb{Z}$ (the finiteness of $\mathbb{Z}/n\mathbb{Z}$ provides a more favorable setting).

Lemma I.1.1.4
Let κ be finite. A probability measure $\mu_n \in \mathcal{M}\left(E_\kappa^{\mathbb{Z}/n\mathbb{Z}}\right)$ is invariant under T on the circle of

length n if

$$\mathsf{Sys}(\mathbb{Z}/n\mathbb{Z},\nu,\mathsf{T}) := \left\{\mathsf{Cycle}_n(x,\mu_n)=0,\ \text{for any } x\in E_\kappa^{\mathbb{Z}/n\mathbb{Z}}\right. \tag{I.11}$$

for

$$\mathsf{Cycle}_n(x,\mu_n) = \sum_{w\in E_\kappa^{\mathbb{Z}/n\mathbb{Z}}} \mu_n(w)\mathsf{T}_{[w|x]} - \mu_n(x)\mathsf{T}_{[x|w]}, \tag{I.12}$$

where $\mathsf{T}_{[w|z]}$ has to be adapted to fit with the structure of $\mathbb{Z}/n\mathbb{Z}$:

$$\mathsf{T}_{[w|z]} = \sum_{[\![a+1,a+L]\!]\subset\mathbb{Z}/n\mathbb{Z}} \mathsf{T}_{[w[\![a+1,a+L]\!]|z[\![a+1,a+L]\!]]}1_{w_j=z_j \text{ for all } j\in(\mathbb{Z}/n\mathbb{Z})\setminus[\![a+1,a+L]\!]}, \tag{I.13}$$

where in this context, $[\![a+1,a+L]\!]$ stands for $(a+1 \mod n,\dots,a+L \mod n)$.

When κ is finite, the existence of a measure μ_n solving the system (I.11) is granted from the theory of finite state space Markov processes.

Again, we disconnect the problem of existence of particle systems with the solution of the algebraic system:

Definition I.1.1.5

For κ finite or infinite, we say that a probability measure $\mu_n\in\mathcal{M}\left(E_\kappa^{\mathbb{Z}/n\mathbb{Z}}\right)$ is cyclically algebraic invariant under T on the circle of length n (we write μ_n is CAI by T on the circle of length n) if it solves $\mathsf{Sys}(\mathbb{Z}/n\mathbb{Z},\nu,\mathsf{T})$ as stated in (I.11).

Invariance and algebraic invariance are equivalent when $\kappa<+\infty$.

The results.

Definition I.1.1.6

▪ For $-\infty<a\le b<+\infty$, a process $(X_k,k\in[\![a,b]\!])$ is said to be a Markov chain on E_κ, or to have a Markov law, if there exists $M:=\left[M_{i,j}\right]_{i,j\in E_\kappa}$, a Markov kernel (we will say also simply kernel), and an initial distribution $\nu\in\mathcal{M}(E_\kappa)$ such that,

$$\mathbb{P}(X_k=x_k,a\le k\le b)=\nu_{x_a}\prod_{j=a}^{b-1}M_{x_j,x_{j+1}},\ \text{for any } x\in E_\kappa^{[\![a,b]\!]}.$$

For short, we will say that X (resp. μ) is a (ν,M)-Markov chain on $[\![a,b]\!]$ (resp. (ν,M)-Markov law) if its kernel is M, and its initial distribution is ν.

Definition I.1.1.7

▪ We will say that a law ρ in $\mathcal{M}(E_\kappa)$ is invariant for M (or for this Markov chain) if $\rho M=\rho$, for ρ seen as a row vector. If the initial distribution is ρ, we say that X is a M Markov chain under (one of) its invariant distribution.

▪ For $\rho\in\mathcal{M}(E_\kappa)$ invariant for M, we call (ρ,M)-Markov chain $(X_k,k\in\mathbb{Z})$ a process indexed by $\mathbb{Z}$ whose finite dimensional distribution are given by $\mathbb{P}(X_k=x_k,a\le k\le b)=$

$\rho_{x_a} \prod_{j=a}^{b-1} M_{x_j,x_{j+1}}$, for any $x \in E_\kappa^{[\![a,b]\!]}$. Its distribution is called (ρ, M)-Markov law.

– A M-Markov law on E_κ is said to be *positive recurrent* if under this kernel, a Markov chain is **positive recurrent** (we will say also that M is positive recurrent).
– If all the $M_{i,j}$'s are positive, we write $M > 0$.

Consider a Markov chain with kernel M on E_κ under its invariant distribution ρ.

Let us define

$$\mathsf{Line}_n^{\rho,M,\mathsf{T}}(x[\![1,n]\!]) := \mathsf{Line}^{\mathbb{Z}}(x[\![1,n]\!], \nu, \mathsf{T}), \text{ for any } x[\![1,n]\!] \in E_\kappa^n \tag{I.14}$$

where $\nu(a[\![1,m]\!]) = \rho_{a_1} \prod_{j=1}^{m-1} M_{a_j,a_{j+1}}$: in words, $\mathsf{Line}_n^{\rho,M,\mathsf{T}}(\cdot)$ is the function which coincides with $\mathsf{Line}^{\mathbb{Z}}(\cdot, \nu, \mathsf{T})$ on E_κ^n when ν is the (ρ, M)-Markov law (see Definition I.1.1.6).

The system of equations $\{\mathsf{Line}_n^{\rho,M,\mathsf{T}} \equiv 0, \text{ for any } n\}$, (as stated in (I.17)) provides the necessary and sufficient algebraic relations between ρ, M and T for the AI of the M-Markov law. This is an infinite system of equations even when E_κ is finite. It is linear in T, with unbounded degree in M.

■ The first goal of this chapter is to produce an equivalent **finite system** of algebraic equations to characterize the invariance of (ρ, M)-Markov law by T when **the set** E_κ **is finite.** The main result is the proof of equivalence of $\{\mathsf{Line}_n^{\rho,M,\mathsf{T}} \equiv 0, \text{ for any } n\}$ with each of several (equivalent) algebraic systems of degree 6 in M and linear in T (Theorem I.2.1.2, and Theorem I.2.1.8, when the range is $L = 2$ and the memory of the Markov chain is $m = 1$). These equivalent systems are finite, and moreover, they can be explicitly solved using some linear algebra arguments (Theorem I.2.5.1): in words, it is possible to decide if a PS with JRM T possesses an invariant Markov law, or to describe the class of all T that do (which provide some applications discussed in Section I.1.1).

- When the cardinality of E_κ is infinite some additional complications arise (Section I.3.2), but some results still hold.
- When M possesses some zero entries, a plurality of algebraic behaviors for these systems of equations (and solutions) makes a global approach probably impossible (Section I.3.4).

■ Similar criteria are developed to characterize product measures $\rho^{\mathbb{Z}}$ invariant by T. In this case the finite representations use equations of degree 3 in ρ and linear in T (Theorem I.2.2.2, when the range $L = 2$).
■ The invariance of the Gibbs distribution with kernel M on the circle $\mathbb{Z}/n\mathbb{Z}$ is also studied, when E_κ is finite. In Theorem I.2.1.2 the equivalence between the invariance of a Gibbs measure (see Definition I.2.1.1) with Markov kernel M on $\mathbb{Z}/n\mathbb{Z}$ for $n = 7$ with the invariance of the (ρ, M)-Markov law (for ρ such that $\rho M = \rho$) on the line $\mathbb{Z}$ is established (Theorem I.2.1.2). Besides, Corollary I.2.1.3 implies that if the Gibbs distribution with kernel M is invariant by T on $\mathbb{Z}/n\mathbb{Z}$ for $n = 7$, then it is also invariant by T on $\mathbb{Z}/n\mathbb{Z}$ for any $n \geq 3$ (when the range is $L = 2$).
■ When considering a PS indexed by the segment $[\![1, n]\!]$, some interactions β^r and β^ℓ with the boundaries are introduced (Section I.2.6). When the range $L = 2$, if a Markov law is invariant for $n \geq 7$ on the segment (with fixed boundaries interactions), then it is invariant on the line (Theorem I.2.6.2). Some relations between invariant measures on the line and on the segment are provided.
■ The 2D case and beyond will be discussed in Theorem I.2.4.1, where a simple necessary and sufficient condition for the invariance of a product measure will be provided (Section I.2.3).
■ The case where T has a larger range L and/or where the invariant distribution is a Markov law with larger memory m is discussed in Section I.3.

Many extensions discussed in Section I.3 to larger range and memory, are proved by the same ideas as those for $L = 2$, with some extra technical complications. We think that the presentation of the proof in the case $L = 2$ is needed in order to make the arguments understandable.

Applications.

As said above, the theorems we provide allow one to decide if there exists a Markov law with kernel M (with memory m) invariant under the dynamics of a PS with a given T. This is done "by explicitly" solving a finite polynomial system with "small degree in M". These kinds of problems are solved using some algebraic tools, for example, the computation of a Gröbner basis (see Section I.4.1), using some Computer algebra systems if needed. The theorems also allow to find pairs (T, M) for which this invariance occurs, and then, to design some PS having a simple known invariance distribution.

Hence, having in hands a simple algebraic characterization of PS admitting invariant Markov law, allows to extend considerably the family of PS for which explicit invariant distributions can be found, and we think that, as illustrated by what we are saying below about hidden Markov distributions, the interest of these results go far beyond invariant Markov laws.

In the sequel, when we say that we use a specific model with general rates, we mean that we let the positive rates as free variables. In addition to the results presented in the preceding section we present here several applications of our work.
▪ In Section I.4.1, we prove that the voter models does not admit any Markov law of any memory as invariant distribution. The general rate version is explored and the parameters for which there exist Markov law invariant on the line are discussed.
▪ In Section I.4.1, the contact process is discussed: we prove that this process does not have a Markov law of any memory $m \geq 0$ as invariant distribution.
▪ In Section I.4.1, the TASEP and some variants are explored: Zero-range type processes, 3 colors TASEP and PushASEP.
– For the zero range type processes we prove that there exists a family of distributions F, such that depending on T, either all the product measures $\rho^{\mathbb{Z}}$ are invariant by T for all $\rho \in F$, or none of them is invariant by T.
– In the general rate 3-color TASEP some sufficient and necessary conditions on T are given so that there exists a Markov law with a positive-entries kernel M that is invariant by T.
– For the PushASEP we explain how some special types of PS with range $L = \infty$ can be transformed and solved with our results.
▪ In Section I.4.1, the stochastic Ising model is analyzed and its well known Markov invariant measure on the line (Gibbs on the cycle) is found based on our results.
▪ The possibility offered by our theorems to find automatically parameters (T, M), say, on the space $E_3 = \{0, 1, 2\}$ (with 3 colors) and $L = 2$ for which the PS with JRM T let the Markov law with kernel M invariant, allows to find some PS on $E_2 = \{0, 1\}$ with 2 colors and $L = 3$ which possesses some hidden Markov chain distributions as invariant distributions, using some projection from E_3 to E_2. As far as we are aware of, this is the first time that a hidden Markov chain is shown to be invariant under a PS on the line. This is discussed in Section I.4.2. We think that this method will allow in the future to find many invariant distribution for PS with 2 colors, or more.
▪ In Section I.4.3, the set of pairs (T, M) for which the Markov law with positive-entries kernel M is invariant under T, in the case $\kappa = 2$ and $L = 2$ is totally explicitly solved. This case corresponds to standard PS on the line, where 1 and 0 are used to model the presence, or absence of particles at each position. Under these assumptions and mass preservation (see Def. I.4.1.4) we prove that the unique Markov kernels that are AI by this type of T's are the i.i.d. measures.

■ In Section I.4.3, the set of pairs (ρ, T) for which the product measure with marginal ρ is invariant under T, in the case $\kappa = 2$ and $L = 2$ is totally explicitly solved.
■ In Section I.4.4 we use our criteria of invariance of product measures under the dynamics of a PS defined on $\mathbb{Z}^2$, to provide many explicit PS admitting product measures as invariant measure.

I.1.2 Some pointers to related papers

The fact that the construction provided in Section I.1.1 defines indeed a process $(\eta^t, t \geq 0)$ can be better viewed on a Poisson point process (this is the so-called graphical representation due to Harris [62] see also Swart's book [108]). For this, consider the Poisson point process Θ on $J \times \mathbb{R}^+$ with intensity

$$I = \sum_{(i,w,w') \in J} \mathsf{T}_{[w|w']} \delta_{(i,w,w')}\, dt,$$

where δ_x is the Dirac measure at x. Equip this set with the partial time order $<$ so that $(i, w_1, w_1', s_1) < (j, w_2, w_2', s_2)$ if $s_1 < s_2$. Denote by $\Theta_t = \{(i, w, w', s) \in \Theta, s \leq t\}$ the events occurring before time t. Define η^t as the image of η^0 by the maps $m_{i,w,w'}$ for $(i, w, w', s) \in \Theta_t$ composed in the time order.

Consider, for $\#E_\kappa < +\infty$ (and this is also valid for $\#E_\kappa = +\infty$ when $\sum_{w,w'} \mathsf{T}_{[w|w']} < +\infty$), and for a fixed time t, the set S_t of j's which have no point of Θ_t in their $L-1$ neighborhood:

$$S_t := \left\{ j\ : \Theta_t \cap \Big([\![j - (L-1), j + (L-1)]\!] \times (E_\kappa^L)^2 \times [0, t] \Big) = \varnothing \right\}.$$

The set S_t is a.s. infinite, and a.s. possesses $+\infty$ and $-\infty$ as accumulation points, so that every $i \in \mathbb{Z}$ is either in S_t or in a finite connected component of $\mathbb{Z} \backslash S_t$. Therefore, a.s. only a finite number of points in Θ_t have affected the value of $\eta^t(i)$, for any i. This implies that a.s. η^t is well defined.

We call such a PS a L-site dependent PS with κ colors ((L, κ)-PS for short).

The Markov generator of this process in the favorable cases, including those for which E_κ is finite (where $E_\kappa^{\mathbb{Z}}$ is compact), is given by (I.4) and it is defined on a large class of functions f, large enough, to conclude that η is a Feller process on the closure of C_0 (the set of functions depending on a finite number of coordinates) (see [108, Section 4]).

When the state space E_κ is infinite, a complication comes since the state on a site may diverge in a finite time. As a matter of fact, in this case, it may happen that some JRM T do not allow to define a time continuous Markov process.

Here are two lists of works related to the present one:

- The problem of well definition, representation and construction of a given (class of) PS. Two main methods are used to prove the existence of a PS: the embedding on a Poisson point process as exposed above (inspired by Harris [62]), or by some means coming from measure theory and functional analysis (Hille-Yosida theorem). See e.g. [83, 108, 71, 4], where proofs of existence and construction can be found in some particular cases). The infinite case is treated, for example, in [83, Chap. IX], [71, 7, 4, 45].
- The computation of invariant distribution(s) of a given PS, or the characterization of its ergodicity (see e.g. [15, 29, 45, 55]).

Other works concern the study of PS out of equilibrium, their speed of convergence, their time to reach a certain state, among others. All these works are not directly related to the present work.

As far as we are aware, the paper whose point of view is the closest to the present work, is [45], in which some conditions for the invariance of product measures are designed, for mass migration processes (see def. I.4.1.5 and below).

We add that this work has been inspired by some similar works on probabilistic cellular automata, where the transition matrices for which simple invariant measures exist, have been deeply investigated, and are at the heart of the theory, [109, 31, 85, 23].

For a general Markov process $\eta = (\eta^t, t \geq 0)$, a probability distribution μ on $E_\kappa^{\mathbb{Z}}$ is said to be *invariant* if $\eta^t \sim \mu$ for any $t \geq 0$, when $\eta^0 \sim \mu$. This can be expressed in terms of the semigroup of the Markov process $(P_t, t \geq 0)$ as

$$\int P_t(f) d\mu = \int f d\mu,$$

for $f \in C(E_\kappa^{\mathbb{Z}})$ (see e.g. [84, Chap. 1]). Using compactness argument, it may be proved that invariant measures exist when E_κ is finite ([84, Prop.1.8]).

When the generator $Gf := \lim_{t \to 0}(P_t f - f)/t$ is well defined on a domain F which contains a class of functions that characterizes the convergence in distribution (which is the case when $E_\kappa < +\infty$), the following characterization is valid ([84, Prop.2.13]): μ is an invariant measure iff μ satisfies

$$\int G f d\mu = 0. \tag{I.15}$$

If well defined the distributions $(\mu^t, t \geq 0)$, where $\mu^t = \mu P_t$ satisfy:

$$\frac{\partial}{\partial t}\mu^t_{[\![n_1,n_2]\!]}(x[\![n_1,n_2]\!]) = \int G \mathbf{1}_{\{w[\![n_1,n_2]\!]=x[\![n_1,n_2]\!]\}} d\mu^t(w) = \mathsf{Line}^{\mathbb{Z}}(x[\![n_1,n_2]\!], \mu^t). \tag{I.16}$$

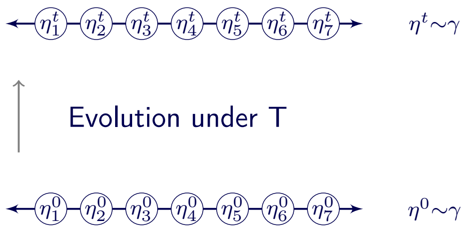
We are interested in invariant distribution for which the LHS in (I.16) is 0, which leads to (I.9) by considering the invariance in t, and then, by replacing μ^t by ν. Hence, the notion of AI is more general than the notion of invariant distribution since it does not require the well-definition of the PS. Nevertheless, when the characterization given by I.15 is valid, invariance and algebraic invariance coincide.

Additional note.

It is important to understand different elements playing a role here:

1. The interacting particle system $\eta = (\eta^t)_{t \in \mathbb{R}^+}$ is a Markovian process on time (when well define), meaning that the whole line is evolving following the Markovian dynamics that we already explain from T.
2. For a fixed t, the process $\eta_t = (\eta^t_k)_{k \in \mathbb{Z}}$ is a process on space, not necessarily Markovian. Nevertheless, we want to find a Markov distribution γ being invariant for T, meaning that there exists M a positive recurrent Markov kernel with ρ an invariant distribution for M (i.e. $\rho M = \rho$) such that for a finite pattern $x \in E_\kappa^{[\![i,k]\!]}$

$$\gamma(x) = \rho_{x_i} \prod_{j=i}^{k} M_{x_j, x_{j+1}}$$



I.2 Main results

The case $L = 1$ being non interesting here, we examine in details the case where the range is $L = 2$, representative of this kind of models as will be seen in Section I.3 where larger ranges will be investigated.

For a given JRM T, the *exit rate out* of $w \in E_\kappa^2$ is defined by

$$\mathsf{T}^{\text{out}}_{[w]} = \sum_{w' \in E_\kappa^2} \mathsf{T}_{[w|w']}.$$

Consider a Markov chain with kernel M, and let ρ be one of its invariant distribution. The equation $\text{Line}_n^{\rho,M,\mathsf{T}}(x[\![1,n]\!]) = 0$ (as defined in (I.14)) rewrites

$$\begin{aligned} &\sum_{\substack{x_{-1},x_0,\\ x_{n+1},x_{n+2} \in E_\kappa}} \sum_{j=0}^{n} \sum_{u,v \in E_\kappa} \mathsf{T}_{[u,v|x_j,x_{j+1}]} \rho_{x_{-1}} \Big(\prod_{\substack{-1 \le k \le n+1\\ k \notin \{j-1,j,j+1\}}} M_{x_k,x_{k+1}} \Big) M_{x_{j-1},u} M_{u,v} M_{v,x_{j+2}} \\ &- \sum_{\substack{x_{-1},x_0,\\ x_{n+1},x_{n+2} \in E_\kappa}} \Big(\rho_{x_{-1}} \prod_{k=-1}^{n+1} M_{x_k,x_{k+1}} \Big) \sum_{j=0}^{n} \mathsf{T}^{\text{out}}_{[x_j,x_{j+1}]} = 0. \end{aligned} \tag{I.17}$$

<u>**Notation:**</u> From now on, when the context is clear we will not write $\sum_{u \in E_\kappa}$, instead we just write $\sum_u$.

From Lemma I.1.1.2, a (ρ, M) Markov law under its invariant distribution is invariant by T on the line when $\text{Line}_n^{\rho,M,\mathsf{T}} \equiv 0$, for all $n \in \mathbb{N}$.

Since the range is $L = 2$, the value of x_0 and x_{n+1} "just outside" $x[\![1,n]\!]$ play a role (they are in the dependence set of $[\![1,n]\!]$, as defined in Def. I.7) we then need to sum on all the possible values of (x_0, x_{n+1}). But, because of the appearance of the pattern $M_{x_i,u} M_{u,v} M_{v,x_{i+3}} \mathsf{T}_{[u,v|x_{i+1},x_{i+2}]}$, it is a bit simpler to consider also additionally the extra values (x_{-1}, x_{n+2}) in the sum even if they are not in the dependence set: these additional terms concern only the representation of the Markov law, and also the fact that ρ is the invariant distribution of M (not the JRM).

We now present the main theorems of the chapter. The proofs that are not given in this section, are postponed to Section I.5.

I.2.1 Invariant Markov laws with positive-entries kernel

In this section, E_κ is finite, and the Markov kernel $M = \left[M_{i,j}\right]_{i,j\in E_\kappa}$ has positive entries. The measure ρ is the invariant law for a Markov chain with kernel M, and is characterized by $\rho = \rho M$.

Additional note.

Since we consider M with positive entries, it is positive recurrent, implying that it possess a unique invariant measure ρ and that for any distribution ρ' on $\mathbb{E}_k$, $\rho' M^n \to \rho$ as $n \to \infty$. We always consider M together with its invariant distribution in the AI setting, this is based on the fact that any position $k \in \mathbb{Z}$ the chain has already run for infinite time, meaning that η_k^t follows the distribution ρ.

Define the normalized version of Line by:

$$\mathsf{NLine}_n^{\rho,M,\mathsf{T}}(x) := \frac{\mathsf{Line}_n^{\rho,M,\mathsf{T}}(x)}{\prod_{j=1}^{n-1} M_{x_j,x_{j+1}}}, \tag{I.18}$$

so that, for $n=1$, and any $x \in E_\kappa$, $\mathsf{NLine}_1^{\rho,M,\mathsf{T}}(x) := \mathsf{Line}_1^{\rho,M,\mathsf{T}}(x)$ and for $n \geq 2$, and any $x \in E_\kappa^n$,

$$\mathsf{NLine}_n^{\rho,M,\mathsf{T}}(x) = \sum_{\substack{x_{-1},x_0,\\ x_{n+1},x_{n+2}}} \left(\rho_{x_{-1}} \prod_{k\in\{-1,0,n,n+1\}} M_{x_k,x_{k+1}}\right) \sum_{j=0}^{n} \mathsf{Z}^{M,\mathsf{T}}_{x[\![j-1,j+2]\!]}, \tag{I.19}$$

with

$$\mathsf{Z}^{M,\mathsf{T}}_{a,b,c,d} := \left(\sum_{(u,v)\in E_\kappa^2} \mathsf{T}_{[u,v|b,c]} \frac{M_{a,u}M_{u,v}M_{v,d}}{M_{a,b}M_{b,c}M_{c,d}}\right) - \mathsf{T}^{\mathsf{out}}_{[b,c]}. \tag{I.20}$$

We will drop the exponents M,T and write $\mathsf{Z}_{a,b,c,d}$ instead when they are clear from the context.

Now, for $u[\![1,\ell]\!]$ a ℓ-tuple of elements of E_κ, denote by

$$\mathsf{Seq}_k(u[\![1,\ell]\!]) = \{u[\![m+1,m+k]\!], 0 \leq m \leq \ell-k\}$$

the multiset[2] "of k-subwords" of $u[\![1,\ell]\!]$ so that, for example

$$\mathsf{Seq}_4(a[\![1,7]\!]) = \{\ a[\![1,4]\!],\ a[\![2,5]\!],\ a[\![3,6]\!],\ a[\![4,7]\!]\ \}.$$

Define the map $\mathsf{Master}_7^{M,\mathsf{T}} : E_\kappa^7 \to \mathbb{R}$ by

$$\mathsf{Master}_7^{M,\mathsf{T}}(a[\![1,7]\!]) = \sum_{w\in\mathsf{Seq}_4(a[\![1,7]\!])} \mathsf{Z}_w - \sum_{w\in\mathsf{Seq}_4(a[\![1,7]\!]^{\{4\}})} \mathsf{Z}_w \tag{I.21}$$

where (following our notation, below the abstract) $a[\![1,7]\!]^{\{4\}} = (a_1,a_2,a_3,a_5,a_6,a_7)$. The map $\mathsf{Master}_7^{M,\mathsf{T}}$ will play an important role in the sequel. Let us expand for once, this compressed notation:

$$\begin{aligned}\mathsf{Master}_7^{M,\mathsf{T}}(a[\![1,7]\!]) &= Z_{a_1,a_2,a_3,a_4} + Z_{a_2,a_3,a_4,a_5} + Z_{a_3,a_4,a_5,a_6} + Z_{a_4,a_5,a_6,a_7}\\ &\quad - Z_{a_1,a_2,a_3,a_5} - Z_{a_2,a_3,a_5,a_6} - Z_{a_3,a_5,a_6,a_7}.\end{aligned}$$

Additional note.

2. A multiset is a set in which elements may have multiplicities ≥ 1

This definition comes in a natural way, when we compare the normalized balanced equations of two words w and w', where w' equals w up to the suppression of a central letter. one word and another, which is the same word with one letter suppressed in the middle.

Now, extend the notion of subwords to $\mathbb{Z}/n\mathbb{Z}$: for $a[\![0,n-1]\!] \in E_\kappa^{\mathbb{Z}/n\mathbb{Z}}$, write

$$\mathsf{Sub}_k^{\mathbb{Z}/n\mathbb{Z}}(a[\![0,n-1]\!]) = \{a[\![m+1 \mod n, m+k \mod n]\!], 0 \le m \le n-1\}$$

for the multiset formed by the n words made of k successive letters of $a[\![0,n-1]\!]$ around $\mathbb{Z}/n\mathbb{Z}$. Define the map $\mathsf{Cycle}_n^{M,\mathsf{T}} : E_\kappa^{\mathbb{Z}/n\mathbb{Z}} \to \mathbb{R}^+$ by

$$\mathsf{Cycle}_n^{M,\mathsf{T}}(x) := \sum_{w \in E_\kappa^{\mathbb{Z}/n\mathbb{Z}}} \left(\prod_{j=0}^{n-1} M_{w_i,w_{i+1} \mod n}\right) \mathsf{T}_{[w|x]} - \left(\prod_{j=0}^{n-1} M_{x_i,x_{i+1} \mod n}\right) \mathsf{T}_{[x|w]}. \quad \text{(I.22)}$$

This formula coincides with $\mathsf{Cycle}_n(x,\mu_n)$ given in Defi. I.1.1.5 for a Gibbs measure with kernel M:

Definition I.2.1.1

A process $(X_k, k \in \mathbb{Z}/n\mathbb{Z})$ indexed $\mathbb{Z}/n\mathbb{Z}$ for some $n \ge 1$ and taking its values in $E_\kappa^{\mathbb{Z}/n\mathbb{Z}}$ is said to have a Gibbs measure with kernel $[M_{i,j}]_{i,j\in E_\kappa}$, a non negative Matrix, if

$$\mathbb{P}(X[\![0,n-1]\!] = x[\![0,n-1]\!]) \quad = \quad \frac{\prod_{j=0}^{n-1} M_{x_j,x_{j+1} \mod n}}{\mathrm{Trace}(M^n)}, \text{ for any } x[\![0,n-1]\!] \in E_\kappa^{[\![0,n-1]\!]}.$$

For short, we will say that X follows the M-Gibbs measure on $\mathbb{Z}/n\mathbb{Z}$.

The Perron-Frobeniüs theorem asserts that if a square matrix A is non negative and irreducible, then A has a real eigenvalue λ larger (or equal, if A is periodic) than the modulus of the other ones, and the corresponding right and left eigenvectors may be chosen with positive entries. We qualify by "main" in the sequel these eigenvectors and eigenvalue. Hence, if M is irreducible, one may suppose w.l.o.g that M is a classical Markov kernel, since $[M'_{i,j}]_{i,j\in E_\kappa} = [M_{i,j}q_i/(cq_j)]_{i,j\in E_\kappa}$, where q is the main right eigenvector of M and c is the corresponding eigenvalue of M, is a Markov kernel which induces the same Gibbs measure as M.

When $\#E_\kappa < +\infty$, a M-Gibbs measure is invariant by T on $\mathbb{Z}/n\mathbb{Z}$ iff $\mathsf{Cycle}_n^{M,\mathsf{T}} \equiv 0$. Again, when $\#E_\kappa = +\infty$, independently of the good definition of the PS with JRM with kernel M, we will say that cyclically algebraic invariant (or CAI) by T on $\mathbb{Z}/n\mathbb{Z}$ when $\mathsf{Cycle}_n^{M,\mathsf{T}} \equiv 0$.

It must be noticed at this point that $\mathsf{Cycle}_n^{M,\mathsf{T}}(x)$ coincides with $\mathsf{Cycle}_n(x,\mu_n)$ as defined in (I.12) where μ_n is the M-Gibbs measure with kernel M. Further for any $n \ge 1$, define the map $\mathsf{NCycle}_n^{M,\mathsf{T}} : E_\kappa^{\mathbb{Z}/n\mathbb{Z}} \to \mathbb{R}$ by

$$\mathsf{NCycle}_n^{M,\mathsf{T}}(x) = \frac{\mathsf{Cycle}_n^{M,\mathsf{T}}(x)}{\prod_{j=0}^{n-1} M_{x_i,x_{i+1} \mod n}}.$$

By inspection, it can be checked that, for $n \ge 3$, diving (I.22) by $\prod_{j=0}^{n-1} M_{x_i,x_{i+1} \mod n}$ gives

$$\mathsf{NCycle}_n^{M,\mathsf{T}}(x) = \sum_{w \in \mathsf{Sub}_4^{\mathbb{Z}/n\mathbb{Z}}(x)} \mathsf{Z}_w, \quad \text{for any } x \in E_\kappa^{\mathbb{Z}/n\mathbb{Z}}. \quad \text{(I.23)}$$

This identity fails for $n = 1$ and $n = 2$.

Also define the map $\mathsf{Replace}_7^{M,\mathsf{T}} : E_\kappa^7 \times E_\kappa \to \mathbb{R}$ by

$$\mathsf{Replace}_7^{M,\mathsf{T}}(a[\![1,7]\!]; a_4') = \sum_{w \in \mathsf{Seq}_4(a[\![1,7]\!])} \mathsf{Z}_w - \sum_{w \in \mathsf{Seq}_4(a_1a_2a_3a_4'a_5a_6a_7)} \mathsf{Z}_w.$$

In fact

$$\mathsf{Replace}_7^{M,\mathsf{T}}(a[\![1,7]\!]; a_4') = \mathsf{NCycle}_7^{M,\mathsf{T}}(a[\![1,7]\!]) - \mathsf{NCycle}_7^{M,\mathsf{T}}(a_1a_2a_3a_4'a_5a_6a_7).$$

It is then the balance in $\mathsf{NCycle}_7^{M,\mathsf{T}}(a[\![1,7]\!])$ when the "central letter" a_4 of a word $a[\![1,7]\!]$ is replaced by a_4'.

Additional note.

> In general Replace is the equation obtained from the difference of invariance equations associated to two words, which differ in one central letter.

A key result of the chapter is the following: the infinite system of equations $\{\mathsf{Line}_n^{\rho,M,\mathsf{T}} \equiv 0, n \geq 1\}$, which by definition is the invariance of the Markov law by T on the line is equivalent to many different finite systems of equations with bounded degree (in M):

Theorem I.2.1.2
Let E_κ be finite and $L = 2$. If $M > 0$ then the following statements are equivalent:

- (i) (ρ, M) is invariant by T on the line.
- (ii) $\mathsf{Replace}_7^{M,\mathsf{T}}(a,b,c,d,0,0,0;0) = 0$ for all $a,b,c,d \in E_\kappa$.
- (iii) $\mathsf{Replace}_7^{M,\mathsf{T}} \equiv 0$.
- (iv) $\mathsf{Master}_7^{M,\mathsf{T}}(a,b,c,d,0,0,0) = 0$ for all $a,b,c,d \in E_\kappa$.
- (v) $\mathsf{Master}_7^{M,\mathsf{T}} \equiv 0$.
- (vi) $\mathsf{NCycle}_n^{M,\mathsf{T}} \equiv 0$ for all $n \geq 3$.
- (vii) $\mathsf{NCycle}_7^{M,\mathsf{T}} \equiv 0$.
- $(viii)$ $\mathsf{NCycle}_7^{M,\mathsf{T}}(a,b,c,d,0,0,0) = 0$ for all $a,b,c,d \in E_\kappa$.
- (ix) There exists a function $\mathsf{W} : E_\kappa^3 \to \mathbb{R}$ such that $\mathsf{Z}_{a,b,c,d}^{M,\mathsf{T}} = \mathsf{W}_{b,c,d} - \mathsf{W}_{a,b,c}$.

Key idea.

> Several implications are direct. The difficult part is to show that Master is enough to recover the complete AI, i.e. that $\mathsf{Line}_n \equiv 0$ for all $n \in \mathbb{N}$. This is a consequence of the Markovian structure and of the structure of the equation which let us go from configurations of length n to configurations of length $n-1$. The consequence that it is enough to test the family with zeroes at the end, comes from a combinatorial structure on the equations.

The implication $(vii) \Rightarrow (vi)$ and $(vii) \Rightarrow (i)$ gives the following Corollary:

Corollary I.2.1.3
Let E_κ be finite. If the Gibbs measure with a positive-entries Markov kernel M is invariant on

$\mathbb{Z}/n\mathbb{Z}$ by T for $n = 7$, then it is also invariant on $\mathbb{Z}/n\mathbb{Z}$ by T for every $n \geq 3$, and the M-Markov law under its invariant distribution is invariant by T on the line.

Remark I.2.1.4
■ The appearance of "0" everywhere in the Theorem is arbitrary. It may be replaced by any constant element of E_κ in the previous statements.
■ The positivity of M is a strong condition whose relaxation entails many difficulties. It is discussed in Section I.3.4.
■ [link with reversibility] The condition

$$\mathsf{T}_{[u,v|b,c]} M_{a,u} M_{u,v} M_{v,d} = M_{a,b} M_{b,c} M_{c,d} \mathsf{T}_{[b,c|u,v]} \text{ for any } a,b,c,d,u,v \in E_\kappa \qquad \text{(I.24)}$$

is equivalent to the fact that PS with JRM T is reversible with respect to the Gibbs measure with kernel M on any cylinder with size ≥ 3. As usual reversibility implies invariance. However, invariance and reversibility are not equivalent even for Gibbs measures: Theorem I.2.1.2 gives the complete picture. In particular, (I.24) implies $\mathsf{Z}^{M,\mathsf{T}}_{a,b,c,d} \equiv 0$, which implies Conditions (ii) to (ix) of Theorem I.2.1.2. The converse does not hold.
■ Further in this chapter, we will state Theorem I.3.1.5 which implies a result somehow stronger that Theorem I.2.1.2 in some conditions: $\mathsf{NCycle}^{M,\mathsf{T}}_n \equiv 0$ for any $n \leq \kappa$ is necessary and sufficient for the Markov chain (ρ, M) to be invariant by T. When the number of colors $\kappa < 7$ this provides a criterion potentially simpler to check than those given in Theorem I.2.1.2.

We state here a theorem which is important in many applications. Consider a PS with JRM T defined on $\mathbb{Z}$, and its analogue on $E_\kappa^{\mathbb{Z}/n\mathbb{Z}}$. For a and b two elements of this last configuration set, b is said to be accessible from a, if $a = b$ or if it is possible to go from a to b using jumps with positive rates. A strict subset S of $E_\kappa^{\mathbb{Z}/n\mathbb{Z}}$ is said to be absorbing, if for any b in $E_\kappa^{\mathbb{Z}/n\mathbb{Z}}$, an element of S is accessible from b, and if $E_\kappa^{\mathbb{Z}/n\mathbb{Z}} \setminus S$ is not accessible from S.

Theorem I.2.1.5
Consider a finite alphabet E_κ with $|\kappa| \geq 2$. Consider T a JRM with range L, such that T is not identically 0.
Suppose that for infinitely many integers n the PS with JRM T possesses an absorbing subset S_n of $E_\kappa^{\mathbb{Z}/n\mathbb{Z}}$, with $\varnothing \subsetneq S_n \subsetneq E_\kappa^{\mathbb{Z}/n\mathbb{Z}}$. Under these conditions, there does not exist any Markov law with memory m, for any m, with full support, invariant by T on the line.

In fact only the case $m = 1$ is a Corollary of Theorem I.2.1.2, the strongest form for general memory $m \geq 1$ and range L is a Corollary of Theorem I.3.1.2 which treats the invariance of Markov law with memory m.

Remark I.2.1.6
■ Notice that if T is identically 0, then all states are absorbing states, then all Markov law are invariant.
■ If the hypothesis of the theorem holds for some fixed n, then the conclusion holds if the memory size m satisfies $m + L \leq n$.

We will use this theorem in Section I.4 for some applications on the contact process and the voter model.

Proof. By Theorem I.2.1.2 (for $m = 1$) or Theorem I.3.1.2 (for $m \geq 1$), if there exists a Markov law with memory m and full support invariant for T on the line, then the same property holds on $\mathbb{Z}/n\mathbb{Z}$ for $n \geq m + L$ for the corresponding Gibbs measure. But the invariance of a full support measure is incompatible with the existence of a non trivial absorbing subset. □

Remark I.2.1.7: Crucial

■ $\text{Master}_7^{M,\mathsf{T}} \equiv 0$ is equivalent to

$$\sum_{w \in \mathsf{Seq}_4(a[\![1,7]\!])} \mathsf{Z}_w = \sum_{w \in \mathsf{Seq}_4(a[\![1,7]\!]^{\{4\}})} \mathsf{Z}_w, \quad \text{for any } a[\![1,7]\!] \in E_\kappa^7. \tag{I.25}$$

This relation allows to see that the LHS of (I.25), does not depend on a_4, and in many places it will allow us to remove "one letter" in linear combinations involving Z: for any words $a[\![1,n]\!]$ with at least $n \geq 7$ letters, for any $4 \leq k \leq n-3$,

$$\sum_{w \in \mathsf{Seq}_4(a[\![1,n]\!])} \mathsf{Z}_w = \sum_{w \in \mathsf{Seq}_4(a[\![1,n]\!]^{\{k\}})} \mathsf{Z}_w.$$

This property is reminiscent to other algebraic properties, as rewriting systems, dependence in a vector space or as relation in the presentation of a group by generators and relations.

■ The fact that $\text{Replace}_7 \equiv 0$ is a necessary condition for the Markov law (ρ, M) to be invariant by T on the line appears naturally since it is the comparison of the balance of the outgoing and incoming rate of two similar words. The sufficiency of this condition is not obvious (see Section I.3.4 for extension when M is not supposed to have positive entries).

There are many links between the systems $\text{NCycle}_n \equiv 0$ for different values of n, here are some of them, which prove that the Markov law with Markov kernel M is invariant by T if (M, T) solves a system of equations with degree 6 in M, linear in T (the system being finite when κ is finite):

Theorem I.2.1.8

The system $\text{NCycle}_7^{M,\mathsf{T}} \equiv 0$ is equivalent to:

(i) $\{\text{NCycle}_6^{M,\mathsf{T}} \equiv 0, \text{NCycle}_5^{M,\mathsf{T}} \equiv 0\}$,

(ii) $\{\text{NCycle}_6^{M,\mathsf{T}} \equiv 0, \text{NCycle}_4^{M,\mathsf{T}} \equiv 0\}$.

Key idea.

> The idea here is to find an equivalent system with the smallest possible degree on M. This is motivated from the fact that (ii), (iv), $(xiii)$ in theorem I.2.1.2 say that there is a subfamily of words that gives a sufficient condition to check invariance, which may be expressed in the diminution of degree.

The proof is given in Section I.A

Remark I.2.1.9

A natural question is: are $\text{NCycle}_6^{M,\mathsf{T}} \equiv 0$ and $\text{NCycle}_7^{M,\mathsf{T}} \equiv 0$ equivalent ? We tested this with a computer for $\kappa = 5$ (by the computation of some Gröbner basis), where the answer turns out to be negative. We will see in the sequel that 7 is the "critical" length of the systems associated to the range $L = 2$. In section I.3 we will give the critical length associated with a general range L.

In the sequel, $W_1 \cdots W_k$ will stand for the word obtained by the concatenation of the words $W_1, \cdots, W_{k-1}$ and W_k.

Remark I.2.1.10: Linearity principle
From Theorem I.2.1.2, if a Markov law with Markov kernel $M > 0$ is invariant by T on the line, then the M-Gibbs measure is invariant in $\mathbb{Z}/n\mathbb{Z}$ for any $n \geq 3$. This is something which can be guessed and proved as follows. Take three words: p, w, and s, the "prefix", the "pattern", and the "suffix". Consider the word $W_n = pw^n s$. If the M-Markov law is invariant by T on the line, then $\mathsf{NLine}^{\rho,M,\mathsf{T}}_{|p|+n|w|+|s|}(W_n) = 0$. But it is easy to see that $\mathsf{NLine}^{\rho,M,\mathsf{T}}_{|p|+n|w|+|s|}(W_n) = (n-1)\mathsf{NCycle}^{M,\mathsf{T}}_{|w|}(w) + O(1)$, so that one infers that $\mathsf{NCycle}^{M,\mathsf{T}}_{k} \equiv 0$ for every $k \geq 3$.

In fact, this remark is also valid for any range L, and even the converse holds (see Theorem I.3.1.5).

I.2.2 Invariant Product measures

Definition I.2.2.1
A process $(X_k, k \in I)$ indexed by a finite or countable set I is said to have the *product distribution* $\mathbf{p}^I$ for a distribution $\mathbf{p}$ on E_κ if the random variables X_k's are i.i.d. and have common distribution $\mathbf{p}$.

Since product measures are special Markov laws, we can use what has been said so far to characterize invariant product measure by T by replacing $M_{i,j}$ by ρ_j in the previous considerations (and rewrite, for example Theorem I.2.1.2 restricted to this special case). But, the "7" appearing everywhere in this theorem is no more relevant for product measure... the crucial length here is "3"! To see this, observe that when $M_{i,j} = \rho_j$, the quantity $\mathsf{Z}^{M,\mathsf{T}}_{a,b,c,d}$ does not depend on (a, d), so that we may set

$$\mathsf{Z}^{\rho,\mathsf{T}}_{b,c} := \mathsf{Z}^{M,\mathsf{T}}_{a,b,c,d} = \sum_{(u,v)\in E_\kappa^2} \mathsf{T}_{[u,v|b,c]} \frac{\rho_u \rho_v}{\rho_b \rho_c} - \mathsf{T}^{\mathsf{out}}_{[b,c]}. \tag{I.26}$$

$\mathsf{Master}^{M,\mathsf{T}}_7$, $\mathsf{Replace}^{M,\mathsf{T}}_7$ and $\mathsf{NCycle}^{M,T}_n(a[\![0,n-1]\!])$ respectively "simplify to"

$$\begin{aligned}
\mathsf{Master}^{\rho,\mathsf{T}}_3(a_0,a_1,a_2) &:= \mathsf{Z}^{\rho,\mathsf{T}}_{a_0,a_1} + \mathsf{Z}^{\rho,\mathsf{T}}_{a_1,a_2} - \mathsf{Z}^{\rho,\mathsf{T}}_{a_0,a_2}, \\
\mathsf{Replace}^{\rho,\mathsf{T}}_3(a_0,a_1,a_2;a_1') &:= \mathsf{Z}^{\rho,\mathsf{T}}_{a_0,a_1} + \mathsf{Z}^{\rho,\mathsf{T}}_{a_1,a_2} - \mathsf{Z}^{\rho,\mathsf{T}}_{a_0,a_1'} - \mathsf{Z}^{\rho,\mathsf{T}}_{a_1',a_2}, \\
\mathsf{NCycle}^{\rho,\mathsf{T}}_n(a[\![0,n-1]\!]) &:= \sum_{j=0}^{n-1} \mathsf{Z}^{\rho,\mathsf{T}}_{a_j,a_{j+1} \mod n} \quad \text{for } n \geq 2.
\end{aligned} \tag{I.27}$$

We have the following analogue of Theorem I.2.1.2, which provides some finite certificate/criteria for the algebraic invariance of product measures.

Theorem I.2.2.2
If E_κ is finite, $L = 2$, and if $\rho \in \mathcal{M}(E_\kappa)$ with support E_κ, then the following statements are equivalent:

(i) $\rho^{\mathbb{Z}}$ is invariant by T on the line.
(ii) $\text{Replace}_3^{\rho,\mathsf{T}}(a,b,0;0) = 0$ for all $a,b \in E_\kappa$.
(iii) $\text{Replace}_3^{\rho,\mathsf{T}} \equiv 0$.
(iv) $\text{Master}_3^{\rho,\mathsf{T}}(a,b,0) = 0$ for all $a,b \in E_\kappa$.
(v) $\text{Master}_3^{\rho,\mathsf{T}} \equiv 0$.
(vi) $\text{NCycle}_n^{\rho,\mathsf{T}} \equiv 0$ for all $n \geq 2$.
(vii) $\text{NCycle}_3^{\rho,\mathsf{T}} \equiv 0$.
(viii) $\text{NCycle}_3^{\rho,\mathsf{T}}(a,b,0) = 0$ for all $a,b \in E_\kappa$.
(ix) There exist a function $\mathsf{W} : E_\kappa \to \mathbb{R}$ such that $\mathsf{Z}_{a,b}^{\rho,\mathsf{T}} = \mathsf{W}_b - \mathsf{W}_a$ for all $a,b \in E_\kappa$.

In fact Theorem I.2.2.2 is not a corollary of Theorem I.2.1.2, but its proof is almost the same.

Remark I.2.2.3: Comparison with detailed balance condition
Consider a probability distribution ρ on E_κ with full support. A natural/folklore sufficient condition for this measure to be invariant by T on the line is the fact that it solves the following system:

$$\rho_b\rho_c\mathsf{T}_{[b,c|u,v]} = \rho_u\rho_v\mathsf{T}_{[u,v|b,c]} \text{ for any}, b,c,u,v \in E_\kappa. \tag{I.28}$$

Summing this over (u,v), one sees that this condition implies $\mathsf{Z}^{\rho,\mathsf{T}} \equiv 0$. Theorem I.2.2.2 applies to these situations since when $\mathsf{Z}^{\rho,\mathsf{T}} \equiv 0$, (ii) to (iii) are clearly satisfied.

The crucial point here is that $\mathsf{Z}^{\rho,\mathsf{T}} \equiv 0$ is just a sufficient condition, not a necessary one (as we will see by providing examples in Section I.4):Theorem I.2.2.2 gives the complete necessary and sufficient conditions.

Remark I.2.2.4
For the sake of simplicity, in Section I.2.2 we restrict ourselves to criteria/properties for invariant of product measures with full support. Nevertheless, contrary to the Markov case, the case of product measures with a smaller support can be also considered without any problem [see Section I.3.3].

"Range 2" on a more general class of graphs. Most of the previous discussions on AI Markov law rely on the geometry of $\mathbb{Z}$, but it turns out that for AI product measures, some of the previous properties still hold when one defines a PS on a more general graph – in the case where it still relies on a JRM with range 2.

Formally, consider a continuous-time Markov process $X = (X_v, v \in V)$ defined on a lattice like $G = \mathbb{Z}^d$ or $(\mathbb{Z}/n\mathbb{Z})^d$. Assume that the pair of states $(\eta(x), \eta(y))$ of two vertices x and y, jumps to the new pair of states (a,b) with rates $p(x,y)\mathsf{T}_{[\eta(x),\eta(y)|a,b]}$ for $p(\cdot,\cdot)$, a translation invariant non negative function (that is $p(u,v) = p(0,v-u)$). By p, the rates also depend on the positions. We suppose that there exists $\mathcal{N} \in \mathbb{N}$ such that $p(x,y) = 0$ if $\|x-y\|_1 \geq \mathcal{N}$ for all $x,y \in G$.

In this case the equilibrium equations for $x(A) \in E_\kappa^A$ for $A \subset G$ finite is

$$\text{Line}^{\rho,\mathsf{T},p}(x(A))$$

$$:= \sum_{\substack{w\in E^{D(A)} \\ w(A)=x(A)}} \sum_{\substack{(i,j)\in D(A)^2 \\ i\in A \text{ or } j\in A}} p(i,j) \left(\sum_{(u,v)\in E_\kappa^2} \frac{\rho_u\rho_v}{\rho_{w_i}\rho_{w_j}} \mathsf{T}_{[u,v|w_i,w_j]} - \mathsf{T}^{\text{out}}_{[w_i,w_j]} \right) \prod_{i\in D(A)} \rho_{w_i}$$

$$= \sum_{\substack{w\in E^{D(A)} \\ w(A)=x(A)}} \sum_{\substack{(i,j)\in\in D(A)^2 \\ i\in A \text{ or } j\in A}} \mathsf{Z}^{\rho,\mathsf{T}}_{w_i,w_j} p(i,j) \prod_{i\in D(A)} \rho_{w_i}$$

where $D(A)$ denotes as before the dependence set, which definition needs to be extended for this type of graphs to $D(A) = \{v \in V : \max\{p_{u,v} + p_{v,u}, u \in A\} > 0\}\}$.

Definition I.2.2.5
We will say that ρ^G is AI by $p\mathsf{T}$ if ρ^G satisfies $\mathsf{Line}_A^{\rho,\mathsf{T},p} \equiv 0$ for all finite $A \subset G$ (again when E_κ is finite and G locally finite, invariance and algebraic invariance are equivalent notions.

Theorem I.2.2.6
Let $\#E_\kappa < +\infty$, $L = 2$ and $\rho \in \mathcal{M}(E_\kappa)$ with full support. Depending on p we have the following equivalences

1. If p is symmetric. ρ^G is invariant by $p\mathsf{T}$ iff $\mathsf{NCycle}_2^{\rho,\mathsf{T}} \equiv 0$.
2. If p is asymmetric. ρ^G is invariant by $p\mathsf{T}$ iff the product measure $\rho^{\mathbb{Z}}$ is invariant by T on the line (see the characterizations in Theorem I.2.2.2).

Key idea.

> The probability p factorizes outside Z in $\mathsf{Line}^{\rho,\mathsf{T},p}$, this gives some space to manipulate them in order to simplify the systems, depending only on the symmetry of p.

Hence, if p is asymmetric, the geometry does not matter since $\mathsf{Z}^{\rho,\mathsf{T}}$ only depends on the states (given by η), and not on the positions.

I.2.3 A glimpse in 2D and beyond

We consider in this part PS indexed by $\mathbb{Z}^d$, whose configuration space is $E_\kappa^{\mathbb{Z}^d}$. We suppose that the JRM instead of being defined (as done in (I.1)) by "the jump rate of size L-subwords" is defined by

$$\mathsf{T} = \left(\mathsf{T}_{[w|w']}\right)_{u,v\in E_\kappa^{\mathsf{HC}[L,d]}}$$

where

$$\mathsf{HC}[L,d] = [\![0, L-1]\!]^d$$

is the hypercube with range L in $\mathbb{Z}^d$ (see discussion in second point of Remark I.2.3.3 for other shapes). For example for $d = 2$, $\mathsf{HC}[2,2]$ is the square Sq formed by the cells $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$. An example of JRM is the following T with all entries equal to 0, except

$$\mathsf{T}\begin{array}{|c|c|}\hline 1\,1 & 0\,0 \\ 0\,1 & 1\,0 \\ \hline\end{array} = 1, \mathsf{T}\begin{array}{|c|c|}\hline 0\,0 & 1\,1 \\ 1\,0 & 0\,1 \\ \hline\end{array} = 1, \qquad \text{(I.29)}$$

meaning that the 2×2 square sub-configurations jumps with rate 1, if they are equal to $\boxed{\begin{matrix}1\,1\\0\,1\end{matrix}}$ or $\boxed{\begin{matrix}0\,0\\1\,0\end{matrix}}$, in which case, the colors of the 4 vertices are flipped.

Formally, replace J defined in (I.2) by

$$J^{(d)} = \left\{(i, w, w'), i \in \mathbb{Z}^d, w \in E_\kappa^{\mathsf{HC}[L,d]}, w' \in E_\kappa^{\mathsf{HC}[L,d]}\right\} = \mathbb{Z}^d \times (E_\kappa^{\mathsf{HC}[L,d]})^2, \tag{I.30}$$

and m by $m^{(d)}$, which again is the family of endofunctions $m_{i,w,w'}^{(d)}$ on the set of configurations defined for any $(i, w, w') \in J^{(d)}$, generalizing naturally the $m_{i,w,w'}$'s defined in (I.3). The corresponding generator is

$$\left(G^{(d)} f\right)(\eta) = \sum_{(i,w,w') \in J^{(d)}} \mathsf{T}_{[w|w']} \left[f(m_{i,w,w'}^{(d)}(\eta)) - f(\eta)\right], \tag{I.31}$$

acting on continuous functions f sufficiently smooth (see discussion below (I.4)). The dynamics of this PS is as follows: starting from a (random or not) configuration $\eta^0 = (\eta_z^0, z \in \mathbb{Z}^d)$, each sub-configuration $(\eta_z^0, z \in h) = u$ indexed by a hypercube h equal to $\mathsf{HC}[L, d]$ up to a translation, is replaced by the sub-configuration with same shape v with rate $\mathsf{T}_{[u|v]}$. When $E_\kappa < +\infty$, this defines a Markov process (see discussion below (I.4)).

Again a measure $\mu \in \mathcal{M}\left(\mathbb{E}_k^{\mathbb{Z}^d}\right)$ is said to be AI by T in $\mathbb{Z}^d$ if its finite dimensional distributions are preserved by T. It is then possible to state the analogue of $\mathsf{Line}^{\mathbb{Z}}$ in these settings: let C be a finite subset of $\mathbb{Z}^d$. Set

$$\mathsf{Line}^{\mathbb{Z}^d}(x(C), \nu) = \sum_{w,z \in E_\kappa^{D(C)}} \left(\nu_{D(C)}(w)\mathsf{T}_{[w|z]} - \nu_{D(C)}(z)\mathsf{T}_{[z|w]}\right) 1_{z(C)=x(C)} \tag{I.32}$$

where $x(C) = (x_c, c \in C)$ is any element of E_κ^C, and where $D(C)$ is the dependence set of C: for any subset F of $\mathbb{Z}^d$, the dependence set of F is

$$D(F) = F - \mathsf{HC}[L, d].$$

Again, for any $w, z \in E_\kappa^{D(C)}$, the global transition rate from w to z is

$$\mathsf{T}_{[w|z]} = \sum_{c \in \mathbb{Z}^2 : (c+h) \cap C \neq \varnothing} \mathsf{T}_{[w(c+h)|z(c+h)]} \mathbf{1}_{w(x)=z(x), \forall x \in D(C) \setminus (c+h)}, \tag{I.33}$$

where $h = \mathsf{HC}[L, d]$. Finally, the normalized version $\mathsf{Line}^{\rho,\mathsf{T}}$ is defined , for any finite domain C by

$$\mathsf{NLine}^{\rho,\mathsf{T}}(x(C)) := \frac{\mathsf{Line}^{\rho,\mathsf{T}}(x(C))}{\prod_{c \in C} \rho_{x(c)}} \text{ for any } x(C) \in E_\kappa^C. \tag{I.34}$$

The first theorem we want to state gives a necessary and sufficient condition for a product measure $\rho^{\mathbb{Z}^d}$ to be invariant by some PS with JRM T. Again, when $E_\kappa < +\infty$ it provides a criterion involving a system composed by **a finite number of equations**. After that, we will explain how to obtain an equivalent system with a much smaller number of equations.

Let C be a finite subset of $\mathbb{Z}^d$ and $D(C)$ its dependence set. The dependence set by definition is a union of hypercubes h with sides L: depending on C, some of them may be included completely in C, some contains some points in C and some points outside. The balance $\mathsf{NLine}^{\rho,\mathsf{T}}(x(C))$ can be decomposed as a sum on these hypercubes. Indeed, using the decomposition of T along simple jump (I.34), one gets

$$\mathsf{NLine}^{\rho,\mathsf{T}}(x(C)) := \sum_{h\subset C} \mathbf{Z}_{x(h)} + \sum_{\substack{h:h\cap C\neq\varnothing \\ h\not\subset C}} \mathbf{Z}^{h\cap C,h}_{x(h\cap C)} \tag{I.35}$$

depending on whether h is totally included in C or not. Here, the geometry of $\mathbb{Z}^d$ appears: when h is not included in C, $h\cap C$ can be (depending on C) any subset of h, and we then need to mark this dependence with the pair $(h\cap C, h)$ as an exponent of $\mathbf{Z}$. A simple analysis on the summation variables and the simplification of the quotient of weights of unchanged colors, give:

$$\mathbf{Z}_{x(h)} = \left(\sum_{y\in E^h_\kappa} \frac{\prod_{c\in y}\rho_c}{\prod_{c\in x(h)}\rho_c} \mathsf{T}_{[y|x(h)]}\right) - \mathsf{T}^{\mathsf{out}}_{[x(h)]} \tag{I.36}$$

and more generally, for h such that $h\cap C\neq\varnothing, h\not\subset C$,

$$\mathbf{Z}^{h\cap C,h}_{x(h\cap C)} = \sum_{w(h)\in E^h_\kappa} \mathbf{Z}_{w(h)} \mathbf{1}_{w(h\cap C)=x(h\cap C)} \prod_{j\in h\setminus C} \rho_{w_j}. \tag{I.37}$$

We said, "more generally" because when $h\subset C$,

$$\mathbf{Z}^{h\cap C,h}_{x(h\cap C)} = \mathbf{Z}^{h,h}_{x(h)} = \mathbf{Z}_{x(h)}. \tag{I.38}$$

When $|E_\kappa|<+\infty$, a product measure $\rho^{\mathbb{Z}^d}$ is AI by T if and only if all the maps $\mathsf{NLine}^{\rho,\mathsf{T}}\equiv 0$. For this, it is not needed that $\mathbf{Z}\equiv 0$ (but it is sufficient):

Theorem I.2.3.1

When $|E_\kappa|<+\infty$, a product measure $\rho^{\mathbb{Z}^d}$ is invariant by T if and only if the two following conditions hold:

(*i*) $\sum_{h:0\in h} \mathbf{Z}^{0,h}_{x(0)} = 0$ where 0 is the origin of $\mathbb{Z}^{(d)}$,

(*ii*) For all subsets C and $C' = C\cup\{c\}$ of $\mathsf{HC}[2L-1,d]$ (where c is a single vertex), and any $x(C')\in E^{C'}_\kappa$,

$$\mathsf{NLine}^{\rho,\mathsf{T}}(x(C')) - \mathsf{NLine}^{\rho,\mathsf{T}}(x(C)) \equiv 0. \tag{I.39}$$

Key idea.

> Here we try to use the one dimension idea, which is to make the invariance equations of a basal family of configurations imply the invariance equations for all patterns by means of increments, i.e. comparison of invariance equations.

as a trivial consequence we get the following condition, weaker than reversibility:

Corollary I.2.3.2

If $E_\kappa<+\infty$ and if $\mathbf{Z}\equiv 0$ then the product measure $\rho^{\mathbb{Z}}$ is invariant by T on $\mathbb{Z}^2$.

Proof. The product measure $\rho^{\mathbb{Z}^d}$ is invariant by T if and only if for at least one sequence $(C_i, i \geq 0)$ of finite subsets of $\mathbb{Z}^d$, such that:
- $C_{i+1} = C_i \cup \{c_{i+1}\}$ (a simple vertex),
- $(C_i, i \geq 0)$ eventually contains an arbitrarily large hypercube,

the property $\mathsf{NLine}^{\rho,\mathsf{T}}(x(C_0)) \equiv 0$, and for all $i \geq 0$, $\mathsf{NLine}^{\rho,\mathsf{T}}(x(C_{i+1})) - \mathsf{NLine}^{\rho,\mathsf{T}}(x(C_i)) = 0$ for any $x \in E_\kappa^{C_{i+1}}$ hold.

Due to (I.36), (I.37) and (I.38), if $C' = C \cup \{c\}$ for a vertex c not in C, the difference $\mathsf{NLine}^{\rho,\mathsf{T}}(x(C')) - \mathsf{NLine}^{\rho,\mathsf{T}}(x(C))$ can be written as a sum of the contributions of the hypercubes h such that $(h \cap C) \neq (h \cap C')$. A simple inspection of the balance in the corresponding sums as expressed in (I.35), gives

$$\mathsf{NLine}^{\rho,\mathsf{T}}(x(C')) - \mathsf{NLine}^{\rho,\mathsf{T}}(x(C)) = \sum_{h: h\cap C' \neq h \cap C} \mathbf{Z}^{h\cap C', h}_{x(h\cap C')} - \mathbf{Z}^{h\cap C, h}_{x(h\cap C)}. \tag{I.40}$$

The theorem states something stronger than the fact that this property holds for all $C' = C_{i+1}, C = C_i$: it suffices that this property holds for those included in $\mathsf{HC}[2L-1, d]$. It remains to say that this last condition comes from (I.40): the difference between the two NLine concerns only the hypercubes h that intersect the new vertex c, and then the union of these hypercubes is included in $\mathsf{HC}[2L-1, d]$. A given union of hypercubes appearing in such a difference can be realized by taking two sets C_{i+1}, C_i included in $\mathsf{HC}[2L-1, d]$. □

Remark I.2.3.3
(i) It is possible to reduce the number of necessary and sufficient conditions in Theorem I.2.3.1 by designing a particular growing sequence (C_i) in such a way that the family $(h, C_i \cap h, C_{i+1} \cap h)$ (up to translation) involved in the right hand side of (I.40) for some i, take only a very small number of values: in $\mathbb{Z}^2$ for 2×2 squares, we can manage to get only 2 (kind of) differences, starting from $C_0 = \{(0,0), (0,1), (1,0)\}$. This is exemplified in Theorem I.2.4.1 and in its proof.
(ii) What has been said so far concerns JRM indexed by hypercubes. If the PS of interests is given using some JRM indexed by some other "shape F", it is still possible to represent such a PS using a JRM indexed by hypercube (by taking a hypercube h large enough to contain F, and by letting the colors in $h \setminus F$ unchanged). However, in $\mathbb{Z}^d$ the number of equations grows rapidly if one uses this kind of expedient. The best thing to do, is to adapt what has been said above to this special shape.

I.2.4 JRM indexed by 2×2 squares in $2D$

Following Remark I.2.3.3, we design a set of necessary and sufficient conditions for invariance of a product measure $\rho^{\mathbb{Z}}$ "less abundant" than those given in Theorem I.2.3.1. We examine this in the 2D case, for a PS with JRM indexed by 2×2 squares, denoted further Sq (as the one given in (I.29)).

Consider the three following sets:

$$\Gamma_0 = \{(0,0), (0,1), (1,0)\}, \quad \Gamma_1 = \Gamma_0 \cup \{(2,0)\}, \quad \Gamma_2 = \Gamma_1 \cup \{(1,1)\}.$$

Theorem I.2.4.1
Let $\kappa < +\infty$. Consider ρ a probability distribution with full support on E_κ and $\mathsf{T} = \left[\mathsf{T}_{[u|v]}\right]_{u,v\in E_\kappa^{\mathsf{Sq}}}$ a JRM indexed by Sq. The measure $\rho^{\mathbb{Z}^2}$ is invariant by T on $\mathbb{Z}^2$ iff the two following conditions hold simultaneously:

(i) $\mathsf{NLine}^{\rho,\mathsf{T}} \equiv 0$ on $E_\kappa^{\Gamma_0}$,

(ii) for any $x \in E_\kappa^{\Gamma_2}$,

$$\mathsf{NLine}^{\rho,\mathsf{T}}(x) - \mathsf{NLine}^{\rho,\mathsf{T}}(x(\Gamma_1)) = 0. \tag{I.41}$$

Key idea.

> Here we just present an example of sufficient condition, since we may choose other set of domains instead of $(\Gamma_i)_{i=0}^2$ in order to have different sufficient conditions.

As a simple corollary: if $\mathsf{NLine}^{\rho,\mathsf{T}} \equiv 0$ on Γ_2 for a ρ with full support then $\rho^{\mathbb{Z}}$ is invariant by T on $\mathbb{Z}$.

Proof. We give a picture based proof, using some representation of computations by pictures.

We insist on the fact that $\rho^{\mathbb{Z}}$ is invariant by T iff all the $\mathsf{NLine}^{\rho,\mathsf{T}}(x(C)) = 0$ for any sub-configuration $x(C) \in E_\kappa^C$, for any subset C of $\mathbb{Z}^2$. As noticed in Theorem I.2.3.1, we just need to prove that for any $s \geq 0$, any square $C = [\![0,s]\!]^2$ is included in a finite domain C' for which $\mathsf{NLine}^{\rho,\mathsf{T}}(x(C')) = 0$ for all $x(C') \in E_\kappa^{C'}$. We will construct a well designed sequence (C_i) satisfying the hypothesis of Theorem I.2.3.1 and containing eventually $[\![0,m]\!]^2$.

Recall formula (I.35), which expresses $\mathsf{NLine}(x(C))$ as a sum of some "Z" indexed by the hypercube included in the dependence domain $D(C)$. In view of Figure I.1 the first hypothesis of Theorem

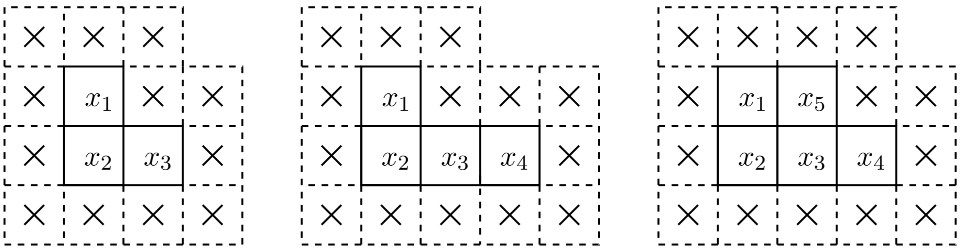


Figure I.1 – Shapes Γ_0, Γ_1 and Γ_2 appearing in Theorem I.2.4.1.

I.2.4.1 says that the sums of these Z over the eight 2×2 squares contained in the first picture of Fig. I.1 is 0. Let us express this by

$$\mathsf{Line}^{\rho,\mathsf{T}}(x(\Gamma_0)) = Z_{\begin{smallmatrix}\times & \times\\ \times & x_1\end{smallmatrix}} + Z_{\begin{smallmatrix}\times & x_1\\ \times & x_2\end{smallmatrix}} + Z_{\begin{smallmatrix}\times & x_2\\ \times & \times\end{smallmatrix}} + Z_{\begin{smallmatrix}\times & \times\\ x_1 & \times\end{smallmatrix}} + Z_{\begin{smallmatrix}x_1 & \times\\ x_2 & x_3\end{smallmatrix}} + Z_{\begin{smallmatrix}x_2 & x_3\\ \times & \times\end{smallmatrix}} + Z_{\begin{smallmatrix}\times & \times\\ x_3 & \times\end{smallmatrix}} + Z_{\begin{smallmatrix}x_3 & \times\\ \times & \times\end{smallmatrix}}.$$

In $Z_{\begin{smallmatrix}y_1 & y_2\\ y_4 & y_3\end{smallmatrix}}$, the variables x_1, x_2, x_3 refers to some fixed specified values and the "x" refers to free variables on which a sum is taken (as in the definition of $\mathbf{Z}_{x(h\cap C)}^{h\cap C,h}$, the "variables in $h \setminus C$" are free variables on which a sum is taken). ☐

Further $\mathsf{Line}^{\rho,\mathsf{T}}(x(\Gamma_1))$ and $\mathsf{Line}^{\rho,\mathsf{T}}(x(\Gamma_2))$ are respectively sums of 10 and 11 such Z: each of these Z must be seen at this stage as indexed by a 2×2 square included in the second and third picture in Fig. I.1 where Γ_1 or Γ_2 are drawn. Many of these Z are common between these structures. It appears then that

$$\begin{aligned}\mathsf{Line}^{\rho,\mathsf{T}}(x(\Gamma_2)) - \mathsf{Line}^{\rho,\mathsf{T}}(x(\Gamma_1)) &= \left(Z_{\boxed{\begin{smallmatrix} x1 & \times \\ x_2 & x_3\end{smallmatrix}}} - Z_{\boxed{\begin{smallmatrix} x_1 & x_5 \\ x_2 & x_3\end{smallmatrix}}}\right) + \left(Z_{\boxed{\begin{smallmatrix} \times & \times \\ x_3 & x_4\end{smallmatrix}}} - Z_{\boxed{\begin{smallmatrix} x_5 & \times \\ x_3 & x_4\end{smallmatrix}}}\right)\\ &+ \left(Z_{\boxed{\begin{smallmatrix} \times & \times \\ x_1 & \times\end{smallmatrix}}} - Z_{\boxed{\begin{smallmatrix} \times & \times \\ x_1 & x_5\end{smallmatrix}}}\right) - Z_{\boxed{\begin{smallmatrix} \times & \times \\ x_5 & \times\end{smallmatrix}}}.\end{aligned}$$

The terms have been assembled to make clear what changes the "appearance" of x_5 in $x(\Gamma_2)$ compared to $x(\Gamma_1)$. Graphically, we use the shortcut given in Figure (I.2). This Picture has to be understood as

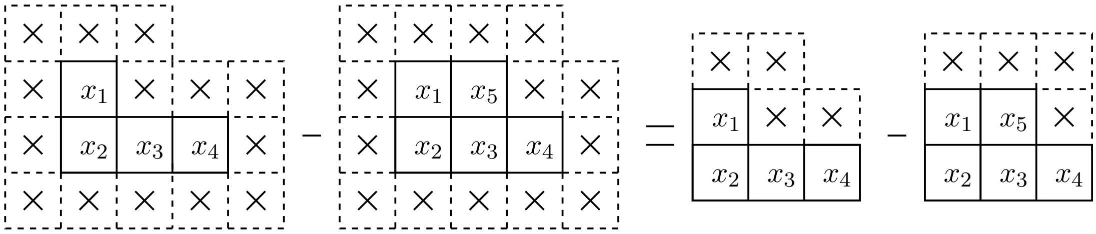


Figure I.2 – Expression of the difference between $\mathsf{Line}^{\rho,\mathsf{T}}(x(\Gamma_2))$ and $\mathsf{Line}^{\rho,\mathsf{T}}(x(\Gamma_1))$.

when one expressed the difference $\mathsf{Line}^{\rho,\mathsf{T}}(x(\Gamma_2))$ and $\mathsf{Line}^{\rho,\mathsf{T}}(x(\Gamma_1))$ by summing on the Z indexed by the squares included in Γ_2 and those included in Γ_1, one gets the same results as if we do the same computation in the small figures in the right hand side in Fig. I.2.

Consider some $n \geq 5$ (to avoid border effects due to the size of Γ_2), and consider the triangle

$$\Delta_n = \{(i,j), 0 \leq i \leq n, 0 \leq j \leq n, 0 \leq i + j \leq n\}.$$

We will show that under the hypothesis of the theorem, for any $x(\Delta_n) \in E_\kappa^{\Delta_n}$, $\mathsf{Line}^{\rho,\mathsf{T}}(x(\Delta_n)) = 0$. For this, we will need the four following steps:
(a) if $\mathsf{Line}^{\rho,\mathsf{T}}(x(\Gamma_0)) = 0$ for any $x(\Gamma_0) \in E_\kappa^{\Gamma_0}$, then $\mathsf{Line}^{\rho,\mathsf{T}}(x(G)) = 0$ if G is the 2×1 or 1×2 domino, or if G is a single vertex (1×1). Indeed, these structures are included in Γ_0, and, for any $G \subset \Gamma_0$, $\mathsf{Line}^{\rho,\mathsf{T}}(x(G)) = 0$ can be obtained by summing $\mathsf{Line}^{\rho,\mathsf{T}}(x(\Gamma_0))$ on the variables which are in $\Gamma_0 \setminus G$.
(b) From (a), we deduce that if L_n is the $n \times 1$ line, then $\mathsf{Line}^{\rho,\mathsf{T}}(x(L_n)) = 0$ for any $x(L_n) \in E_\kappa^{L_n}$. The graphical proof of this property is drawn on Fig. I.3. A single argument is needed: the set of 2×2 square contributions that do not vanish is the same in the right and left hand side.
(c) We now, extend the construction of this row L_n by adding a single vertex y just above the right-most element, getting a new shape L'_n as represented in the top-left picture in Fig. I.4. The graphical proof provided in Fig. I.4 allows to prove that $\mathsf{Line}^{\rho,\mathsf{T}}(x(L'_n)) = 0$ using the nullity of Line on $E_\kappa^{L_n}$, $E_\kappa^{\Gamma_0}$ and on dominoes.
(d) The argument given in (c) is independent from the fact that L_n was the first row. Since the difference $\mathsf{Line}^{\rho,\mathsf{T}}(x(L'_n)) - \mathsf{Line}^{\rho,\mathsf{T}}(x(L_n))$ does not involve the square below row at level 1 (say), if we "complete" both L_n and L'_n by the same fixed row at level 0, the difference $\mathsf{Line}^{\rho,\mathsf{T}}(x(L'_n)) - \mathsf{Line}^{\rho,\mathsf{T}}(x(L_n))$ would be unchanged. Hence, if two structures S and S' are equal up to a given row at level h, and differs only because S' possesses an additional point just above the leftest position of this row, then we still have $\mathsf{Line}^{\rho,\mathsf{T}}(x(S')) - \mathsf{Line}^{\rho,\mathsf{T}}(x(S)) = 0$.

Adding a single vertex above the left-most point of the top-most row is a construction which does not allows to pass from L_n to Δ_n. We still need an elementary growing trick to allow to put some new vertices at the right of the top-most vertex in L'_n to complete the second row (in fact, we will

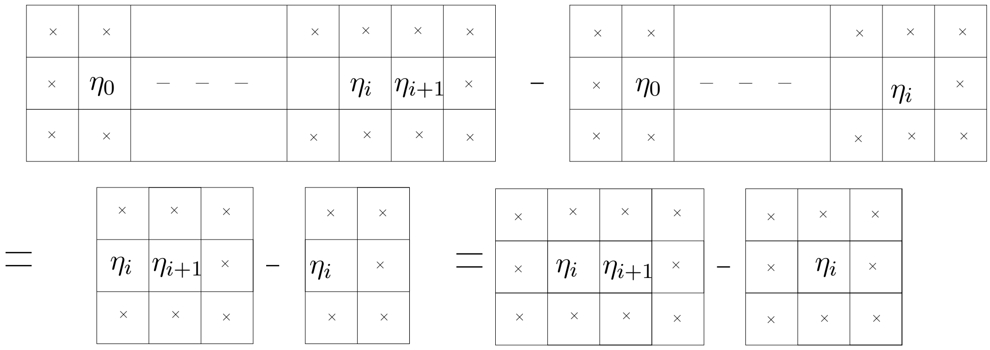


Figure I.3 – Representation of the geometry of the computation of $\text{NLine}^{\rho,\mathsf{T}}(x(L_{i+1})) - \text{NLine}^{\rho,\mathsf{T}}(\eta(L_i))$: each sum has to be taken on the set of 2×2 squares included in the drawn rectangles. All squares appearing in both pictures simplify and then the geometry of the summation reduces to that of the second line. In the third line, some squares are added, but since they correspond to the same contributions, this is allowed.

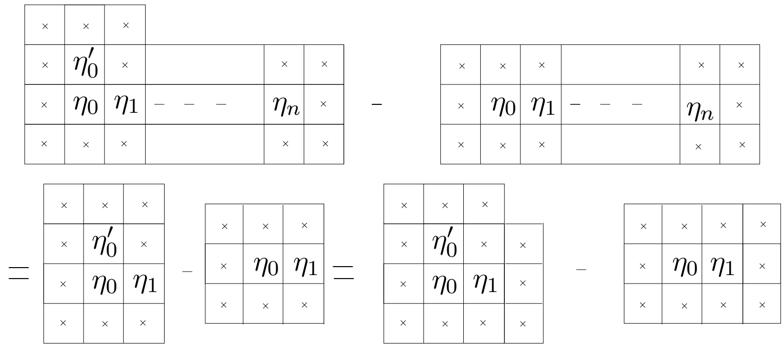


Figure I.4 – The equation resulting of the addition of a vertex above the leftest corner of the upest row.

construct a new row with one vertex less than L_n, leading iteratively to Δ_n): a slight generalization of Figure I.2 that do the job, and the graphical computation is represented in Fig. I.5:

I.2.5 How to explicitly find invariant Markov law or invariant product measures on the line?

In real applications, often T is given, and the need is to find a Markov kernel M so that the M-Markov law is AI by T. Let us call

$$S_j(\mathsf{T}) = \{\text{Markov kernel } M \ : M > 0, \text{NCycle}_j^{M,\mathsf{T}} \equiv 0\}.$$

When $|E_\kappa| < +\infty$, by Theorem I.2.1.2, to find such M amounts to finding $S_7(\mathsf{T})$ (which can be empty). The algebraic system $\text{NCycle}_7^{M,\mathsf{T}} \equiv 0$ is huge even when κ is small, and then quite difficult to solve: many equations of degree 6 in M (by Theorem I.2.1.8) and linear in T. From Theorem I.2.1.2, we know that if the M-Markov law is invariant by T, $S_7(\mathsf{T}) \subset S_3(\mathsf{T})$. It turns out that computing $S_3(\mathsf{T})$ can be done (see Theorem I.2.5.1), and then, in practice, these solutions can be tested in $\text{NCycle}_7^{M,\mathsf{T}}$ afterwards.

To find T when M is given so that T preserves the M-Markov law is a linear algebra problem

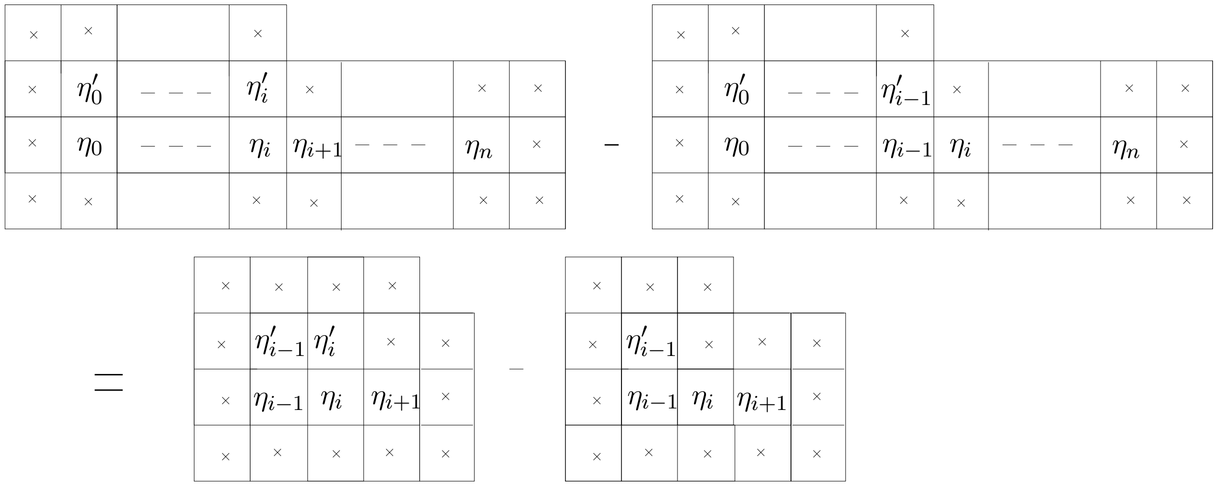


Figure I.5 – For this formula, observe that in the left hand side, the result is unchanged if, instead of taking a first specified row $y[\![0,n]\!]$ one takes unspecified values x, since the squares involving any values of the first row vanishes. The right hand side is 0 because of the second hypothesis of the theorem

since e.g. $\mathsf{Cycle}_7^{M,\mathsf{T}}(a,b,c,d,0,0,0) = 0$ is a linear system in T; the set of solutions is a convex set. Notice that if for a fixed M some tools of linear algebra are used to find the T's solution of e.g. $\mathsf{Cycle}_7^{M,\mathsf{T}} \equiv 0$, then an additional work of identification of non negative solutions is needed.

Computation of $S_3(\mathsf{T})$

Assume T is given, and let us determine $S_3(\mathsf{T})$. Setting

$$\nu_{a,b,c} := \frac{M_{a,b}M_{b,c}M_{c,a}}{\mathsf{Trace}(M^3)}, \text{ for every } a,b,c \in E_\kappa, \tag{I.42}$$

the equation $\mathsf{Cycle}_3^{M,\mathsf{T}}(a,b,c) = 0$ is equivalent to

$$\left\{\begin{array}{l} \sum_{(u,v)} \left(\nu_{c,u,v}\mathsf{T}_{[u,v|a,b]} + \nu_{a,u,v}\mathsf{T}_{[u,v|b,c]} + \nu_{b,u,v}\mathsf{T}_{[u,v|c,a]}\right) \\ = \nu_{a,b,c}\left(\mathsf{T}^{\mathsf{out}}_{[a,b]} + \mathsf{T}^{\mathsf{out}}_{[b,c]} + \mathsf{T}^{\mathsf{out}}_{[c,a]}\right) \end{array}\right. \tag{I.43}$$

This is a linear system in ν, therefore it can be solved by means of linear algebra. If no positive solution ν exists, then $S_3(\mathsf{T}) = \varnothing$. Assume that a positive solution ν exists. Define for any $a,b \in E_\kappa$ the row matrices $L_{a,b}$, the square matrices N_a, and the vector R:

$$N_a = \left[\frac{\nu_{a,x,y}\,\nu_{a,a,a}^{1/3}}{\nu_{a,x,a}}\right]_{x,y\in E_\kappa}, \tag{I.44}$$

$$L_{a,b} = \left[\nu_{a,b,x}, x \in E_\kappa\right], \tag{I.45}$$

$$R = {}^t\left[1, x \in E_\kappa\right]. \tag{I.46}$$

For each a, take the pair of left and right eigenvector ($\ell = \ell_a, r = r_a$) with positive entries of N_a corresponding to the main eigenvalue (notion defined below Def. I.2.1.1), normalized so that $\|\ell_a\|_1 = \ell_a R = 1$, and $r_a\ell_a = 1$. Recall the considerations just above (I.43).

Theorem I.2.5.1
Let $\#E_\kappa < +\infty$, $L = 2$ and ν be a given probability measure on E_κ^3, invariant under rotation, and solving (I.43). If there exists a positive recurrent M-Markov law such that (I.42) holds then all the matrices $(N_x, x \in E_\kappa)$ possess the same main eigenvalue λ.
In case of existence of a positive recurrent Markov kernel M solving (I.42), M is unique and is characterized together with its invariant distribution ρ by

$$\rho_a M_{a,b} = \frac{L_{a,b} r_a}{\lambda^3}. \tag{I.47}$$

Key idea.

> Here we use the cyclic structure of ν to discover the measure associated to configurations in cycles of arbitrary length. We conclude the explicit expression for M by taking the limit in the size of cycles.
> **Tools:** Perron-Frobenius, algebra on cycles.

Remark I.2.5.2
We don't know if the fact that the matrices $(N_x, x \in E_\kappa)$ possess the same main eigenvalue λ implies that there exists a Markov kernel M such that (I.42) holds.

An algorithm to compute $S_3(\mathsf{T})$:
– search the set of probability measures ν solving (I.43),
– for each element of this set (which is moreover invariant by rotation), check if the corresponding N_x's possess the same main eigenvalues λ,
– if yes, compute M using (I.47),
– if this M satisfies (I.42), then add it to the set $S_3(\mathsf{T})$.

Another point of view on the uniqueness of M: The system of equations $\mathsf{Master}_7^{M,\mathsf{T}}$, $\mathsf{NCycle}_n^{M,\mathsf{T}}$ are linear in the $\mathsf{T}_{[a,b|c,d]}$'s, and linear in the rational fractions of the family

$$\mathbf{F} := \left(F_{\substack{(a,u,v,d)\\(b,c)}} := \frac{M_{a,u}M_{u,v}M_{v,d}}{M_{a,b}M_{b,c}M_{c,d}} \right)_{a,b,c,d,u,v \in E_\kappa}, \tag{I.48}$$

since $\mathsf{Z}^{M,\mathsf{T}}_{a,b,c,d}$ has this property. Finding M satisfying $\mathsf{Master}_7^{M,\mathsf{T}} \equiv 0$ for a given T can be done in two steps: first, solve the system of linear equations $\mathsf{Master}_7^{M,\mathsf{T}} \equiv 0$ with the vector $\mathbf{F}$ as unknown variable, and then when $\mathbf{F}$ is found, search if there exists a Markov kernel M which satisfies (I.48). The second step is algebraically the most difficult since (I.48) is a cubic system in M for a given $\mathbf{F}$, nevertheless, we have:

Theorem I.2.5.3
Given $\mathbf{F}$, there exists at most one positive recurrent Markov kernel M solving (I.48).

Additional note.

> We will prove something stronger, namely: for a given sequence $\left(F_{\substack{(a,u,v,a)\\(b,c)}} \right)_{a,b,c,u,v \in E_\kappa}$ with $d = a$,

there is at most one M satisfying (I.48); this amounts to: the measures (Gibbsian or not) solving $\text{Cycle}_3 = 0$ will suffice to find the candidate Gibbsian measures with kernel M if any.

The proof is provided in Section I.B.

Finding the set of invariant product measures

Let T be given and $|E_\kappa| < +\infty$. Now we explore some necessary and/or sufficient conditions for the existence of product measures invariant by T on the line. Define the symmetric version of T by

$$\mathsf{S}_{[a,b|c,d]} = \mathsf{T}_{[a,b|c,d]} + \mathsf{T}_{[b,a|d,c]}.$$

Theorem I.2.5.4
Let $\#E_\kappa < +\infty$, $L = 2$ and $\rho \in \mathcal{M}(E_\kappa)$ with full support.
(i) If the product measure $\rho^{\mathbb{Z}}$ is invariant by T, then $\rho^{\mathbb{Z}}$ is also invariant by S.
(ii) The product measure $\rho^{\mathbb{Z}}$ is invariant by S (or any symmetric JRM S) on the line iff $\mathsf{Z}^{\rho,\mathsf{S}} \equiv 0$.

Proof. (i) The set of JRM that preserve a given invariant distribution is a cone. Now, the product measure $\rho^{\mathbb{Z}}$ is preserved by "space" reversibility: if $\rho^{\mathbb{Z}}$ is invariant by the JRM T then it is also invariant by T' defined by $\mathsf{T}'_{[a,b|c,d]} = \mathsf{T}_{[b,a|d,c]}$.
(ii) When the product measure $\rho^{\mathbb{Z}}$ is invariant by S, then $\text{NCycle}_4^{\rho,\mathsf{S}} \equiv 0$ by Theorem I.2.2.2 which implies $\text{NCycle}_4^{\rho,\mathsf{S}}(a,b,a,b) = 2\left(Z_{a,b}^{\rho,\mathsf{S}} + Z_{b,a}^{\rho,\mathsf{S}}\right) = 4Z_{a,b}^{\rho,\mathsf{S}} = 0$ for any $a, b \in E_\kappa$, and then $\mathbb{Z}^{\rho,\mathsf{S}} \equiv 0$. Conversely, if $Z^{\rho,\mathsf{T}} \equiv 0$, by all the criteria of Theorem I.2.2.2, the product measure $\rho^{\mathbb{Z}}$ is invariant by S on the line. □

Hence, **to know if there exist some product measures invariant by some given** T, one can proceed as follows:
(a) compute S,
(b) solve the equation $\mathbb{Z}^{\rho,\mathsf{S}} \equiv 0$ with unknown S (a pretreatment, can consist to replace in $Z^{\rho,\mathsf{S}}$, each occurrence of $\rho_x\rho_y$ by $\rho_{x,y}$ in order to get a linear equation in the vector $(\rho_{u,v}, (u,v) \in E_\kappa^2))$. After that, it remains to check if indeed $\rho_{u,v}$ can be written under the the form $\rho_u\rho_v$ (notice that in this case $\rho_u = \sqrt{\rho_{u,u}^2}$).
(c) If (b) provides no solution, then no product measure are invariant under T. If (b) provides some solutions, they are candidate to be invariant by T, and it remains to check if whether $\text{Cycle}_3^{\rho,\mathsf{T}} \equiv 0$ or not.

I.2.6 Models in the segment with boundary conditions

In general when one defines a PS on $\mathbb{Z}$ or the segment $[\![1,n]\!]$, where a special behavior at the boundary of the domain is forced.

Definition I.2.6.1
A probability measure $\gamma_n \in \mathcal{M}\left(E_\kappa^{[\![1,n]\!]}\right)$ is said to be AI by $\mathsf{T}^{[\![1,n]\!]}$ on the segment $[\![1,n]\!]$ if it

solves the following system:

$$\mathsf{Sys}([\![1,n]\!],\gamma_n,\mathsf{T}^{[\![1,n]\!]}) \;:=\; \left\{\mathsf{Line}^{[\![1,n]\!],B}(x)=0,\;\text{for any } x\in E_\kappa^{[\![1,n]\!]},\right.$$

where for an extra B in the notation denote the presence of a boundary

$$\mathsf{Line}^{[\![1,n]\!],B}(x)=\sum_{w\in E_\kappa^{[\![1,n]\!]}}\gamma_n(w)\mathsf{T}^{[\![1,n]\!]}_{[w|x]}-\gamma_n(x)\mathsf{T}^{[\![1,n]\!]}_{[x|w]},$$

Recalling that $\mathsf{T}_{[w|z]}$ is the induced jump rate on an interval defined in (I.8), we define $\mathsf{T}^{[\![1,n]\!]}$ as the sum of this induced jump rate to which we add some boundary effects at the left and at the right of the segment given by some jump rate matrices β^ℓ and β^r with range $L-1$:

$$\begin{aligned}\mathsf{T}^{[\![1,n]\!]}_{[w|z]} &= \mathsf{T}_{[w|z]}\\ &+ \beta^\ell\left[w[\![1,L-1]\!],z[\![1,L-1]\!]\right]\mathbf{1}_{w_j=z_j,\forall j\in[\![L,n]\!]}\\ &+ \beta^r\left[w[\![n-(L-2),n]\!],x[\![n-(L-2),n]\!]\right]\mathbf{1}_{w_j=z_j,\forall j\in[\![1,n-(L-1)]\!]}.\end{aligned}$$

We go on focusing on AI Markov law here. Take again M a Markov kernel with positive entries, and ρ its invariant distribution. Define

$$\mathsf{NLine}^{M,B}(x[\![1,n]\!]):=\mathsf{Line}^{M,B}(x[\![1,n]\!])/(\rho_{x_1}\prod_{i=1}^{n-1}M_{x_i,x_{i+1}})$$

where ρ is the unique element of $\mathcal{M}(E_\kappa)$ such that $\rho M=\rho$. For $n\geq 3$, a simple computation shows (see if needed the forthcoming Section I.5.1), that $\mathsf{NLine}_n^{M,B}(x[\![1,n]\!])=$

$$\begin{aligned}&\sum_{j=2}^{n-2}\mathsf{Z}_{x[\![j-1,j+2]\!]}\\ &-\beta^{\mathsf{out},\ell}_{x_1}-\mathsf{T}^{\mathsf{out}}_{[x_1,x_2]}+\sum_{u_1,u_2}\frac{\rho_{u_1}M_{u_1,u_2}M_{u_2,x_3}}{\rho_{x_1}M_{x_1,x_2}M_{x_2,x_3}}\left(\mathsf{T}_{[u_1,u_2|x_1,x_2]}+\beta^\ell_{u_1,x_1}\mathbf{1}_{u_2=x_2}\right)\\ &-\beta^{\mathsf{out},r}_{x_n}-\mathsf{T}^{\mathsf{out}}_{[x_{n-1},x_n]}+\sum_{u_{n-1},u_n}\frac{M_{x_{n-2},u_{n-1}}M_{u_{n-1},u_n}}{M_{x_{n-2},x_{n-1}}M_{x_{n-1},x_n}}(\mathsf{T}_{[u_{n-1},u_n|x_{n-1},x_n]}+1_{u_{n-1}=x_{n-1}}\beta^r_{u_n,x_n}).\end{aligned}$$

Theorem I.2.6.2
Let E_κ be finite, $L=2$ and M be a Markov kernel with positive entries on E_κ. If for some $n_0\geq 7$ the (ρ,M)-Markov law is invariant by $(\beta^r,\beta^\ell,\mathsf{T})$ on $[\![1,n]\!]$ for $n=n_0$ and for $n=n_0+1$, then:

- the (ρ,M)-Markov law is invariant by T on the line,
- the (ρ,M)-Markov law is invariant by $(\beta^r,\beta^\ell,\mathsf{T})$ on $[\![1,n]\!]$ for any $n\geq n_0$.

Key idea.

This is again a consequence of the increments in the invariance equations.

Proof. To prove the first point: By Theorem I.2.1.2, it suffices to prove that $\mathsf{Master}_7^{M,\mathsf{T}}\equiv 0$. So assume that $\mathsf{NLine}^{M,B}_{n_0+1}\equiv 0$ and $\mathsf{NLine}^{M,B}_{n_0}\equiv 0$, and observe that for any $x\in E_\kappa^{n_0+1}$,

$$\mathsf{NLine}^{M,B}_{n_0+1}(x)-\mathsf{NLine}^{M,B}_{n_0}(x^{\{4\}})=\mathsf{Master}_7^{M,\mathsf{T}}(x([\![1,7]\!])),\tag{I.49}$$

(the boundary terms cancel out) Now, to prove the second point, it suffices to observe that (I.49) still holds if one replaces n_0 by a larger integer, so that one can infer from the nullity of $\text{NLine}^{M,B}_{n_0+1}$ and $\text{NLine}^{M,B}_{n_0}$ the nullity of $\text{Master}^{M,\mathsf{T}}_7$, and after that of all $\text{NLine}^{M,B}_{n}$ for $n \geq n_0$. □

Theorem I.2.6.3
Let E_κ be finite, $L = 2$ and M be a Markov kernel with positive entries on E_κ. If the (ρ, M)-Markov law is invariant by T on the line, then there exist two vectors β^r and β^ℓ such that the (ρ, M)-Markov law is invariant by $(\beta^r, \beta^\ell, \mathsf{T})$ on $[\![1, N]\!]$, for any $N \geq 1$.

Key idea.

> Invariance on the line gives a natural flow equilibrium of creation and destruction on the segment in such a way that some consistent parameters can be found in the boundary to preserve the invariance.

Proof. Suppose that a (ρ, M)-Markov law is invariant by T on the line. The key point in the proof if that, if $(X_0, \cdots, X_{n+1})$ has the (ρ, M)-Markov law under its invariant distribution, then $(X_1, \cdots, X_n)$ has also the (ρ, M)-Markov law under its invariant distribution. Hence, it is possible to build explicitly β^ℓ and β^r in such a way they emulate the exterior effects of the segment $[\![1, N]\!]$. It suffices then to take simply

$$\begin{cases} \beta^\ell_{z,a} &= \left(\sum_{u,v} \rho_u M_{u,z} \mathsf{T}_{[u,z|v,a]}\right) / \rho_z \\ \beta^r_{x_n,a} &= \sum_{v,b} \mathsf{T}_{[x_n,v|a,b]} M_{x_n,b}. \end{cases}$$
□

Remark I.2.6.4
What is done in this section is a bit related to the matrix ansatz used by Derrida & al. [32] in order to find and describe the invariant distribution μ_n of the TASEP on a segment $[\![1, n]\!]$, in the sense that it relies on a telescopic scheme.

I.3 Extension to larger range, memory, dimension, etc.

I.3.1 Extension of Theorem I.2.1.2 to larger range and memory

The case $L > 2$ can be treated as the case $L = 2$ has been treated, with some adjustments. Also the case of AI Markov law with memory $m > 1$ can be managed. We discuss both extensions simultaneously here. A first change concerns the "7" which played a special role in Theorem I.2.1.2 which will be replaced by

$$\mathsf{h} = 4m + 2L - 1. \tag{I.50}$$

As usual, a Markov chain with Markov kernel M and memory $m \geq 0$, is a process $(X_k, k \geq k_0)$ (for some k_0) whose distribution is characterized by

$$\mathbb{P}\left(X_j = x_j \mid (X_{j-i} = x_{j-i}, i \geq 1)\right) = M_{x[\![j-m,j]\!]},$$

for $j - m \geq k_0$, and an initial distribution $\mu \in \mathcal{M}(E^m_\kappa)$, the distribution μ of $(X_{k_0}, \cdots, X_{k_0+m-1})$. The Markov kernel M is a matrix with size $\kappa^m \times \kappa$ with non negative entries, such that, for any $x \in E^m_\kappa$, $\sum_{y \in E_\kappa} M_{xy} = 1$. We call such a Markov kernel, a Markov kernel with memory m.

We let $\mathsf{Line}_n^{\rho,M,\mathsf{T}}(x[\![1,n]\!])$ be the equation $\mathsf{Line}^{\mathbb{Z}}(x[\![1,n]\!],\nu)$ where ν is the M-Markov law with memory m and JRM T (we may use the same notation as before, since in the case $(L,m)=(2,1)$ we recover the same definition as before). The equation $\mathsf{Line}_n^{\rho,M,\mathsf{T}}(x[\![1,n]\!])=0$ rewrites:

$$\begin{aligned} 0 &= \sum_{w\in E_\kappa^{[\![-(L-1),n+L-1]\!]}} \sum_{j=-L+2}^{n} 1_{w_k=x_k,k\in[\![1,n]\!]\setminus[\![j,j+L-1]\!]} \sum_{u\in E_\kappa^L} \mu_{G(w,u,j)} \mathsf{T}_{[u|x[\![j,j+L-1]\!]]} \\ &- \sum_{w\in E_\kappa^{[\![-(L-1),n+L-1]\!]}} \sum_{j=-L+2}^{n} \mathsf{T}^{\mathsf{out}}_{[w[\![j,j+L-1]\!]]}\mu(w)\mathbf{1}_{w[\![1,n]\!]=x[\![1,n]\!]}, \end{aligned}$$

for $G(w,u,j)$ being the word w in which $w[\![j,j+L-1]\!]$ has been replaced by u:

$$\begin{aligned} G[w,u,j] &= w[\![-(L-1),j-1]\!]\,u\,w[\![j+L,n+L-1]\!], \\ \mu(w[\![1,N]\!]) &= \rho_{w[\![1,m]\!]}\prod_{j=1}^{N-m} M_{w[\![j,j+m]\!]} \end{aligned}$$

and where ρ is the invariant distribution of the Markov kernel M.

Remark I.3.1.1
Mimicking what has been done in (I.17), and explained below we may write a variant of this formula, by summing on w with index set enlarged, by taking $w\in E_\kappa^{[\![-q,n+L-1]\!]}$ with $q=L-1+m$ (and summing on these words), keeping unchanged the sum on j, in such a way that in the representation of $\mu(w)$ there are no intersection of indices between those involved in ρ and in T.

We also extend the definition of $\mathsf{NLine}^{\rho,M,\mathsf{T}}$ to the present case,:

$$\mathsf{NLine}_n^{\rho,M,\mathsf{T}}(x[\![1,n]\!]) =: \frac{\mathsf{Line}^{\rho,M,\mathsf{T}}(x[\![1,n]\!])}{\prod_{j=1}^{n-m} M_{x[\![j,m+j]\!]}}. \tag{I.51}$$

The quantity which plays the role of Z in these settings is:

$$\mathsf{Z}_{a[\![1,m]\!],b[\![1,L]\!],c[\![1,m]\!]} = \sum_{u[\![1,L]\!]\in E_\kappa^L} \mathsf{T}_{[u[\![1,L]\!]|b[\![1,L]\!]]}\prod_{j=1}^{m+L}\frac{M_{w'[\![j,j+m]\!]}}{M_{w[\![j,j+m]\!]}} - \mathsf{T}^{\mathsf{out}}_{[b[\![1,L]\!]]}, \tag{I.52}$$

where $\mathsf{T}^{\mathsf{out}}_{[u[\![1,L]\!]]} = \sum_{v[\![1,L]\!]\in E_\kappa^L}\mathsf{T}_{[u[\![1,L]\!]|v[\![1,L]\!]]}$ and

$$\left\{\begin{array}{lcl} w &=& a[\![1,m]\!]\,b[\![1,L]\!]\,c[\![1,m]\!], \\ w' &=& a[\![1,m]\!]\,u[\![1,L]\!]\,c[\![1,m]\!]. \end{array}\right.$$

The quantity which will play the role of "4" as in (I.23) is

$$\mathsf{s} = 2m+L = (\mathsf{h}+1)/2.$$

We extend the definition of NCycle_n for $n\geq m+1$:

$$\mathsf{NCycle}_n^{M,\mathsf{T}}(x[\![1,n]\!]) = \sum_{w\in\mathsf{Sub}_{\mathsf{s}}^{\mathbb{Z}/n\mathbb{Z}}(x[\![1,n]\!])} \mathsf{Z}_w;$$

for any $x \in E_\kappa^{\mathsf{h}}$ and $y \in E_\kappa$, extend $\mathsf{Master}^{M,\mathsf{T}}$ and $\mathsf{Replace}^{M,\mathsf{T}}$ by:

$$\begin{aligned}\mathsf{Master}_{\mathsf{h}}^{M,\mathsf{T}}(x) &= \sum_{w\in \mathsf{Seq}_{\mathsf{s}}(x)} \mathsf{Z}_w - \sum_{w\in \mathsf{Seq}_{\mathsf{s}}(x^{\{\mathsf{s}\}})} \mathsf{Z}_w,\\ \mathsf{Replace}_{\mathsf{h}}^{M,\mathsf{T}}(x;y) &= \sum_{w\in \mathsf{Seq}_{\mathsf{s}}(x)} \mathsf{Z}_w - \sum_{w\in \mathsf{Seq}_{\mathsf{s}}(x[\![1,\mathsf{s}-1]\!]\, y\, x[\![\mathsf{s}+1,\mathsf{h}]\!])} \mathsf{Z}_w.\end{aligned}$$

In words, the second sum ranges on the subwords of size s of w with the central letter removed in the case of $\mathsf{Master}^{M,\mathsf{T}}$ and changed in the case of $\mathsf{Replace}^{M,\mathsf{T}}$. For $(L,m)=(2,1)$ we recover the standard definition of $\mathsf{Master}_7^{M,\mathsf{T}}$ and $\mathsf{Replace}_7^{M,\mathsf{T}}$. Here is the main result of this section:

Theorem I.3.1.2
Let E_κ be finite and $L \geq 2$. If $M > 0$ with memory $m \in \mathbb{N}\cup\{0\}$ then the following statements are equivalent:

- (i) (ρ, M) is invariant by T on the line.
- (ii) $\mathsf{Replace}_h^{M,\mathsf{T}}(a[\![1,\mathsf{s}]\!]\, 0^{\mathsf{s}-1}; 0) = 0$ for all $a[\![1,\mathsf{s}]\!] \in E_\kappa^{\mathsf{s}}$.
- (iii) $\mathsf{Replace}_h^{M,\mathsf{T}} \equiv 0$.
- (iv) $\mathsf{Master}_h^{M,\mathsf{T}}(a[\![1,\mathsf{s}]\!]\, 0^{\mathsf{s}-1}) = 0$ for all $a[\![1,\mathsf{s}]\!] \in E_\kappa^{\mathsf{s}}$.
- (v) $\mathsf{Master}_h^{M,\mathsf{T}} \equiv 0$.
- (vi) $\mathsf{NCycle}_n^{M,\mathsf{T}} \equiv 0$ for all $n \geq m + L$.
- (vii) $\mathsf{NCycle}_h^{M,\mathsf{T}} \equiv 0$.
- $(viii)$ $\mathsf{NCycle}_h^{M,\mathsf{T}}(a[\![1,\mathsf{s}]\!]\, 0^{\mathsf{s}-1}) = 0$ for all $a[\![1,\mathsf{s}]\!] \in E_\kappa^{\mathsf{s}}$.
- (ix) There exists a function $\mathsf{W}^{m,L} : E_\kappa^{\mathsf{s}-1} \to \mathbb{R}$ such that $\mathsf{Z}_{a[\![1,\mathsf{s}]\!]}^{M,\mathsf{T}} = \mathsf{W}^{m,L}(a[\![1,\mathsf{s}-1]\!]) - \mathsf{W}^{m,L}(a[\![2,\mathsf{s}]\!])$.

Remark I.3.1.3
The constraint $n \geq m + L$ in (vi) comes from the fact that, if $n < m + L$, the cyclic structure imposes the repetition of some letters in the product of M's inside Cycle. This fact is reflected in Theorem I.2.1.2 (vi) and Theorem I.2.2.2 (vi), where a product measure is seen as a Markov law with memory $m = 0$.

Remark I.3.1.4

(a) A given PS may be represented in several different ways using different JRM: the PS with jump rate $\mathsf{T}_{[0|1]} = \mathsf{T}_{[1|0]} = 1$ with range 1, can be represented using as JRM $\mathsf{T}_{[a,b|1-a,b]} = \mathsf{T}_{[a,b|a,1-b]} = 1/2$ for any $a \neq b, a, b \in \{0,1\}$ instead, on the line and on any $\mathbb{Z}/n\mathbb{Z}$ for $n \geq 2$. In Theorem I.3.1.2, we do not assume that the smallest possible range has been used, but there is a price to pay to use a representation with JRM with a non minimal range since the equations provided by Theorem I.3.1.2 are more numerous, and have a larger degree in M.

(b) The previous point may lead to think that it could be a good idea to represent any PS with a JRM with range 2, which is always possible, by changing the alphabet: if T has range $L > 2$, then by taking the map which sends the set of configurations $\mathbb{E}_\kappa^{\mathbb{Z}}$ onto $A^{\mathbb{Z}}$ where the alphabet $A = \mathbb{E}_\kappa^{L-1}$ by sending $\eta \in E_\kappa^Z$ on to $(\eta'_j, j \in \mathbb{Z})$ where

$$\eta'_j = [\eta_{j+x}, 1 \leq x \leq L-1],$$

that is by rewriting η as a sequence of overlapping subwords on size $L-1$, then one can express on this new space the JRM thanks to a jump rate of range 2. However, since our theorem allows to characterize the invariant Markov law with some fixed memory m, **with full support** they are not suitable to characterize invariant Markov law for η' (since consecutive states η'_j and η'_{j+1} must be consistent, that is the suffix of η'_j must coincide with the prefix of η'_{j+1}).

The proof of Theorem I.3.1.2 is a bit more complex that that of Theorem I.2.1.2 (Section I.C).

Equivalence between $\mathsf{NCycle}_n^{M,\mathsf{T}} \equiv 0$ for small n's and invariant of a M-Markov law on the line. Theorem I.2.1.2 which states that $\mathsf{Cycle}_n^{M,\mathsf{T}} \equiv 0$ for any n is equivalent to $\mathsf{Master}_7^{M,\mathsf{T}} \equiv 0$ is valid only when $L = 2$ and $m = 1$, but the fact that $\mathsf{NCycle}_n^{M,\mathsf{T}} \equiv 0$ for every "small n" is equivalent to the invariance of the M-Markov law on the line, is true in all generality, and can be proved using arguments that are interesting by their own:

Theorem I.3.1.5
Let $\kappa < +\infty$, $L < +\infty$. For a Markov kernel M with memory m and positive entries to be invariant by T, it is necessary and sufficient that $\mathsf{NCycle}_n^{M,\mathsf{T}} \equiv 0$ for any $n \leq \kappa^m$.

Key idea.

We use that the alphabet is finite to show by the pigeonhole principle that any configuration of length bigger than κ^m has a repeated pattern and then it is close to a Cycle equation plus some small order term that we show to be zero.
Tools: Pigeonhole principle, total variation distance.

The proof is given in Section I.D.

I.3.2 The case $E_\kappa = \mathbb{N}$ (that is $\kappa = \infty$)

Here we will consider Markov kernels with positive entries (the case with possibly zero entries is discussed in Section I.3.4). The main problem in the case $\kappa = +\infty$ is that the sums defining Line are now infinite series and therefore some conditions need to be satisfied in order to rearrange terms as done in the proofs, for example, to write Z. The first problems come from the infinitesimal generator (see (I.4)) which may fail to have an interesting domain, in other words, in general, it does not define a Markov process. But even if we jump directly to the AI considerations a second problem arise: it is no more clear that $\mathsf{Line}_n^{\rho,M,\mathsf{T}} \equiv 0$ and $\mathsf{NLine}_n^{\rho,M,\mathsf{T}} \equiv 0$ are equivalent. The series appearing in both members of (I.17) are composed with positive terms. It is necessary and sufficient that each of them converges for Line to be well defined. If each of them converges, Fubini's theorem ensures that we can rearrange globally their terms as wished. Hence, we have under this condition

$$\left(\mathsf{Line}^{\rho,M,\mathsf{T}} \equiv 0\right) \Rightarrow \left(\mathsf{NLine}^{\rho,M,\mathsf{T}} \equiv 0\right). \tag{I.53}$$

The problem is that it is often the converse which is needed, since all criteria we gave rely on Master, NLine, NCycle. When $\#E_\kappa < +\infty$,

$$\left(\mathsf{NLine}^{\rho,M,\mathsf{T}} \equiv 0\right) \Rightarrow \left(\mathsf{Line}^{\rho,M,\mathsf{T}} \equiv 0\right), \tag{I.54}$$

but, when κ is infinite, when a pair (M,T) solving $\mathsf{NLine}^{\rho,M,\mathsf{T}} \equiv 0$ is found, (I.54) must be checked.

The following proposition gives a sufficient condition for the validity of both (I.54) and (I.53).

Proposition I.3.2.1
Assume that $\kappa \in \mathbb{N} \cup \{+\infty\}$, and M is a Markov kernel with positive entries. If

$$\begin{cases} C_1 & := & \sup_{a,b,c,d \in E_\kappa} \sum_{u,v \in E_\kappa} \frac{M_{a,u}M_{u,v}M_{v,d}}{M_{a,b}M_{b,c}M_{c,d}} \mathsf{T}_{[u,v|b,c]} < \infty \\ C_2 & := & \sup_{b,c \in E_\kappa} \mathsf{T}^{\mathsf{out}}_{[b,c]} < \infty, \end{cases}$$

then NLine_n and Line_n as defined in (I.10) and (I.19) are well defined, and satisfy (I.18), and then (I.53) and (I.54) are satisfied.

Proof. Following the discussion above, we verify that under the hypothesis above, the series arising in each term of (I.17) are absolutely convergent. For this notice that it suffices to replace the sign "minus" by "plus" in (I.17) and to bound it by

$$\leq (n+1)(C_1 + C_2)\rho_{x_1} \prod_{k=1}^{n-1} M_{x_k,x_{k+1}}.$$

Hence, if C_1 and C_2 are finite, the sums in $\mathsf{Line}^{\rho,M,\mathsf{T}}$ are well defined and can be rearranged.

In the same way, the positive and negative contributions in (I.19) can be separated and each of them converge absolutely. The conclusion follows. □

Theorem I.3.2.2
Assume that $\kappa \in \mathbb{N} \cup \{+\infty\}$. If the three following conditions holds:
- (I.53) and (I.54) hold,
- M has positive entries,
- M is positive recurrent,

then the conclusion of Theorem I.2.1.2 holds.

Proof. The positive recurrence ensures that the Markov kernel M admit an invariant distribution ρ with full support, from what the proof of Theorem I.2.1.2 can be proved as in the finite case. □

Remark I.3.2.3
The additional assumption of positive recurrence is a natural condition for several reasons. The first one is, in the definition of $\mathsf{Line}^{\rho,M,\mathsf{T}}$, the need of an initial distribution for the Markov chain. When several invariant distributions for M exist (or if none exists), everything is more complex, as discussed in Section I.3.4.

One way to see the appearance of multiple AI Markov laws is to consider two continuous-time Markov processes X^t and Y^t respectively on $A^{\mathbb{Z}}$, and $B^{\mathbb{Z}}$ with $A \cap B = \varnothing$. With them, one may construct a continuous-time Markov process Z^t which coincides with X^t and Y^t if the starting configurations are in $A^{\mathbb{Z}}$, and $B^{\mathbb{Z}}$, respectively by defining:

• for $u \in A^2$, $\mathsf{T}^v_{[Z|u]} = \mathsf{T}^v_{[X|u]}$ for $v \in A^2$, and 0 if $v \notin A^2$
• for $u \in B^2$, $\mathsf{T}^v_{[Z|u]} = \mathsf{T}^v_{[Y|u]}$ for $v \in B^2$, and 0 if $v \notin B^2$
• and if u is not in $A^2 \cup B^2$, choose any value for $\mathsf{T}^{(v_1,v_2)}_{[Z|u_1,u_2]}$. In this case, the set of configurations $A^{\mathbb{Z}}$ and $B^{\mathbb{Z}}$ do not communicate; if both X^t and Y^t possess a AI Markov law, then Z^t possess several invariant Markov laws, including those that are mixture of these. Theorem I.2.1.2 and all its criteria do not allow to characterize this kind of invariant measures.

I.3.3 Invariant product measures with a partial support in E_κ.

We discuss here an iff criterion to show the invariance distribution of a product measures $\nu^{\mathbb{Z}}$ with support S strictly included in E_κ. The idea to get some criteria is just to discard the set $E_\kappa \setminus S$ which should not be reachable from S if an invariant distribution with support S exists:

Consider T a JRM on E_κ, and let S be a strict (non empty) subset of E_κ and ν a measure with support S. Assume that for any $u, v, a, b \in S$

$$\big(\nu_u \nu_v > 0, \mathsf{T}_{[u,v|a,b]} > 0\big) \Rightarrow \nu_a \nu_b > 0 \tag{I.55}$$

and interpret this condition as: if the word w' is obtained from the word $w \in S^{\mathbb{Z}}$ by a jump with positive rate, then w' must be in the support of $\nu^{\mathbb{Z}}$. This implies that the restriction T′ of T to S defined by

$$\mathsf{T}'_{[a,b|c,d]} = \mathsf{T}_{[a,b|c,d]} \text{ for } a, b, c, d \in S$$

has the following property: the PS on $E_\kappa^{\mathbb{Z}}$ (resp. $S^{\mathbb{Z}}$) with JRM T (resp. T′) coincide if starting from a measure ν with support in S. The following theorem is a direct consequence of this fact:

Theorem I.3.3.1
Let $|E_\kappa| < +\infty$. A product measure $\nu^{\mathbb{Z}}$ with support $S = \mathrm{Supp}(\nu) \subset E_\kappa$ is invariant by T on the line, for T a JRM on E_κ iff (I.55) holds as well as any of the equivalent conditions listed in Theorem I.2.2.2 holds within S.

I.3.4 Invariant Markov distributions with MK having some zero entries.

In Theorem I.3.3.1 is discussed the invariance of a product measure which has a partial support in E_κ, and in fact, our criteria apply to this situation up to a simple restriction of the state space.

The same kind of conditions can be imagined for a M-Markov law satisfying $M_{i,j} > 0$ for $i, j \in S$, and such that for any $i \in S$, $\sum_{j \in S} M_{i,j} = 1$, meaning that the states in S just communicate with other states in S. If

$$\forall a, b \in S, \mathsf{T}_{[a,b|u,v]} > 0 \Rightarrow u, v \in S,$$

then, the JRM T can be restricted to S. Denoting by T′ this restriction, the criteria we have (Theorem I.2.1.2) allows to decide if the M-Markov law is invariant by T′ on $S^{\mathbb{Z}}$. Under these conditions, everything is then somehow trivial, since $S^{\mathbb{Z}}$ is close under the action of the jumps with positive rate.

The general case is much more complicated!
Consider a general Markov kernel $M = (M_{i,j})_{i,j \in E_\kappa}$. Consider the directed graph $G = (E_\kappa, E)$ whose vertex set is the alphabet E_κ and the edge set is $E = \{(i,j) : M_{i,j} > 0\}$. Consider the strongly connected components $(\mathcal{C}_j, j \in J)$ of this graph, where J is a set of indices. Starting from any point $v \in E_\kappa$, the Markov chain $(X_n, n \geq 0)$ with kernel M will eventually reach one of these strongly connected components $\mathcal{C}_j$ and will stay inside a.s., forever. The invariant distributions of M naturally decomposes as a mixture of the invariant distributions $\rho^{(j)}$, where $\rho^{(k)}$ is the invariant distribution of M on $\mathcal{C}_k$.

The strongly connected components do not communicate, then, one may partition the vertex sets E_κ along these connected components. The Markov chain on each of this connected component is irreducible and can be treated separately: the fact that one of them is invariant by T does not interfere with the fact that the "other sub-Markov chains" have the same property or not.

The property of being irreducible does not mean that E is the complete graph and some $M_{i,j}$'s can still be 0 in this case. It may also happen that M is periodic, meaning that again, it may exist several invariant distributions with the Markov kernel M (for example, equal up to a translation, alternating between even and odd states).

Again, the range considered here is $L = 2$, and some adjustments need to be made in the next considerations if $L > 2$. Consider an irreducible Markov chain $(X_n, n \geq 0)$ with kernel M. Its invariant distribution has full support on E_κ. Let

$$\mathsf{Supp}_n = \left\{ x[\![1,n]\!] \in E_\kappa^n : \rho_{x_1} \prod_{j=1}^{n-1} M_{x_j, x_{j+1}} > 0 \right\}.$$

be the support of the distribution of n consecutive positions of this Markov chain. A necessary condition for the (ρ, M)-Markov law to be invariant by T on the line is the following *local preservation condition* **(LPC)**:

$$\text{if } (a,b,c,d) \in \mathsf{Supp}_4 \text{ and if } \mathsf{T}_{[b,c|u,v]} > 0 \text{ then } (a,u,v,d) \in \mathsf{Supp}_4.$$

If $x[\![1,n]\!]$ belongs to Supp_n then all its subwords $x[\![m,m+3]\!]$ with 4 letters are in Supp_4. Assume that T possesses the LPC, then the (ρ, M)-Markov law is AI by T if for any $x[\![1,n]\!] \in \mathsf{Supp}_n^M$, $\mathsf{Line}^{\rho,M,\mathsf{T}}(x[\![1,n]\!]) = 0$. Under the LPC, we may still pass from $\mathsf{Line}^{\rho,M,\mathsf{T}}(x[\![1,n]\!])$ to $\mathsf{NLine}^{\rho,M,\mathsf{T}}(x[\![1,n]\!])$ by dividing by $\prod_{i=1}^{n-1} M_{x_i,x_{i+1}}$ as far as $x[\![1,n]\!] \in \mathsf{Supp}_n^M$. Besides, $\mathsf{Z}_{a,b,c,d}^{M,\mathsf{T}}$ is still well defined for $(a,b,c,d) \in \mathsf{Supp}_4$.

Now, solving $\mathsf{NLine}_n^{\rho,M,\mathsf{T}} \equiv 0$ **cannot at all be done according to the same lines as before**, since one cannot compare simply $\mathsf{NLine}_n^{\rho,M,\mathsf{T}}(x[n])$ with $\mathsf{NLine}_n^{\rho,M,\mathsf{T}}(x[n]^{\{k\}})$ (with a suppressed letter) simply, since $x[n] \in \mathsf{Supp}_n \not\Rightarrow x[n]^{\{k\}} \in \mathsf{Supp}_{n-1}$.

It turns out that for a general JRM T, the support of an (algebraic) invariant distribution possesses its own combinatorial structure, which may be really complex.
Indeed T may have some combinatorial properties with a flavor reminiscent to group theory: it is possible to design some JRM T which preserves several non communicating subsets of $E_\kappa^{\mathbb{Z}}$, for example the subset of words $w = (w_i, i \geq \mathbb{Z})$ satisfying $w_i + w_{i+1} \in 17\mathbb{Z} \cup \mathbf{19}\mathbb{Z}$ for all i's (such a property holds for any JRM T satisfying : for any (x,y) such that $x + y \in 17\mathbb{Z} \cup \mathbf{19}\mathbb{Z}$, $\mathsf{T}_{[x,y|x',y']} > 0 \Rightarrow x' + y' \in 17\mathbb{Z} \cup \mathbf{19}\mathbb{Z}$. It is also possible to imagine and design invariant Markov laws with a much more complex support. In any case, the characterization of the set of pairs (M, T) such that

a M-Markov law is invariant is quite complex, and all the tools we used to prove Theorem I.2.1.2 fail. In few words, this happens because it is no longer possible to compare the balance for two close words: for instance, the word obtained from the removal of a letter of a word in the support may not belong to it – and the number of letters to remove in order to go back to the support is a parameter of the system, and of the initial word; it is in general not constant, and unbounded.

I.3.5 Matrix ansatz

The matrix ansatz is a clever method used by Derrida & al. [32] in order to find and describe the invariant distribution μ_n of TASEP on a segment $[\![1,n]\!]$. The matrix ansatz is now widely used (see Blythe and Evans for a survey [15], Crampe & al. [29], and Corteel & al. [27] for its use in relation with combinatorial enumeration) : under some conditions on T (details are given below) it allows to express the invariant distribution μ_n with some explicit finite or infinite matrices. We present their ideas in the following paragraphs, but instead of just focusing on TASEP, we will indicate the most general settings in which it applies. We propose a presentation slightly different than theirs, in order to make more apparent what is general, and what is specific. Before starting, we state a (folklore) representation lemma for $\mathcal{M}\left(E_\kappa^{[\![1,n]\!]}\right)$.

Lemma I.3.5.1

Let $n \geq 1$ and $\kappa \in \mathbb{N} \cup \{+\infty\}$. For any $\mu_n \in \mathcal{M}\left(E_\kappa^{[\![1,n]\!]}\right)$, there exists a one-line matrix L, a one-column matrix R, and some square matrices $(A_x, x \in E_\kappa)$ such that

$$\mu_n(x) = LA_{x_1}\cdots A_{x_n}R, \quad \text{for any } x \in E_\kappa^{[\![1,n]\!]}. \tag{I.56}$$

The matrices $R, L, (A_x, x \in E_\kappa)$ can be taken with non negative entries.

Here, and thereafter, $\prod_{i=1}^n A_{x_i}$ stands for the matrix product $A_{x_1}\cdots A_{x_n}$ in this order.

Proof. Let $X[\![1,n]\!]$ be a process with distribution μ_n. Write

$$\mathbb{P}(X[\![1,n]\!] = x[\![1,n]\!]) = \mathbb{P}(X_1 = x_1)\prod_{i=2}^{n}\mathbb{P}(X[\![1,i]\!] = x[\![1,i]\!] \mid X[\![1,i-1]\!] = x[\![1,i-1]\!])$$

and then (I.56) holds if one takes:
– L indexed by $\bigcup_{j\geq 1} E_\kappa^j$, $L(w)$ being 1 if $|w| = 0$, that is at the entry corresponding to the empty word $\mathcal{E}$, 0 otherwise,
– R indexed by $\bigcup_{j\geq 1} E_\kappa^j$, $R(w)$ being 1 for every word $w \in \bigcup_{j\geq 1} E_\kappa^j$,
– and if, for each y, the matrix A_y is the matrix $A_y(w,w')$ indexed by the (all) pairs of words (w,w') both with sizes $\leq n$ such that:
• $A_y(w,w') = 0$ if w is not the prefix of w' with $|w'|-1$ letter,
• and for any $1 \leq i \leq n$, any $w[\![1,i]\!] \in E_\kappa^i$,

$$A_y(w[\![1,i-1]\!], w[\![1,i]\!]) = \mathbf{1}_{w_i=y}\mathbb{P}(X[\![1,i]\!] = w[\![1,i]\!] \mid X[\![1,i-1]\!] = w[\![1,i-1]\!]).$$

When $i = 1$, this has to be understood as $A_y(\mathcal{E}, a) = \mathbb{P}(X_1 = a)$, where a is a letter. □

From this proof one sees that when $\#E_\kappa < +\infty$, there exists a matrix representation of μ_n with finite matrices A_x, L, R, but these matrices can also be chosen independently from n if the μ_j's are compatible. In this case the proof provides infinite matrices even if in some cases finite matrices would do the job (for example, when μ_n is the distribution of i.i.d. random variables, or a MD): the representation (I.56) is not unique even at the matrix size level.

Consider a segment $[\![1,n]\!]$ and a continuous Markov process $X^t[\![1,n]\!]$ taking its values in $E_\kappa^{[\![1,n]\!]}$ for which is searched an invariant measure μ_n. The approach of Derrida & al. (for TASEP) consists in searching a representation of the invariant measure proportional to

$$f_n(x[\![1,n]\!]) = LA_{x_1}\cdots A_{x_n}R, \tag{I.57}$$

which is a "free of charge" assumption as granted by Lemma I.3.5.1, for some matrices L, A_x, R satisfying some additional constraints. Choosing proportionality instead of exactness is not crucial if one works at a fixed n – since one may divide L by a constant – but this provides a degree of freedom to get a relation between μ_n and μ_{n-1}.

Starting from $\mu_n^{t=0}$ the distribution proportional to f_n, write

$$c_n\frac{d}{dt}\mu_n^t(x[\![1,n]\!]) = L\left(\sum_a \beta^\ell[a,x_1]A_a - \sum_a \beta^\ell[x_1,a]A_{x_1}\right)A_{x_2}\cdots A_{x_n}R \tag{I.58}$$

$$+ \sum_{j=1}^{n-1} L\left(A_{x_1}\cdots A_{x_{j-1}}\right)O_{x_j,x_{j+1}}\left(A_{x_{j+2}}\cdots A_{x_n}\right)R \tag{I.59}$$

$$+ LA_{x_1}\cdots A_{x_{n-1}}\left(\sum_a \beta^r[a,x_n]A_a - \sum_a \beta^r[x_n,a]A_{x_n}\right)R, \tag{I.60}$$

in which c_n is the total mass of f_n, and where

$$O_{c,d} = \sum_{a,b} \mathsf{T}_{[a,b|c,d]}A_aA_b - \mathsf{T}_{[c,d|a,b]}A_cA_d. \tag{I.61}$$

What has been done here is the commutation of the linear differential operator with the matrix product. This may fail when the matrices (L, A_x, R) are infinite, but this can be checked at the end, when some matrices A_k have been found. The invariance of μ_n is granted by $\frac{d}{dt}\mu_n^t(x[\![1,n]\!]) = 0$. The type of this equation is the following

$$y_\ell + t_1 + \cdots + t_{n-1} + y_r = 0, \tag{I.62}$$

where the t_i's, y_r and y_l are functions and the equality is for all possible entries. Letting $s_i = s_0 + t_1 + \cdots + t_i$, one has $s_i - s_{i-1} = t_i$ so that (I.62) rewrite

$$y_\ell - s_0 + s_{n-1} + y_r = 0. \tag{I.63}$$

Hence writing t_i under the form of a difference '$t_i = s_i - s_{i-1}$" is always possible, totally general, and is just characterized by the choice of s_0.

The idea of Derrida & al. is to search the matrices (A_x) that provides a telescopic scheme "at the level of the inner sum", searching solutions (A_x) that satisfies moreover the quadratic equation (in the coefficients of the A_i's)

$$O_{c,d} = -\lambda_c A_d + \lambda_d A_c \tag{I.64}$$

since in this case the sum (I.59) reduces to

$$-\lambda_{x_1} f_{n-1}(x_2, \cdots, x_n) + \lambda_{x_n} f_{n-1}(x_1, \cdots, x_{n-1})$$

a sufficient condition on (L, A_x, R) for f_n to be an invariant measure is simply, in conjunction with (I.64),

$$\begin{cases} L\left(\sum_a \beta^\ell[a, x_1] A_a - \sum_a \beta^\ell[x_1, a] A_{x_1}\right) = \lambda_{x_1} L \\ \left(\sum_a \beta^r[a, x_n] A_a - \sum_a \beta^r[x_n, a] A_{x_n}\right) R = -\lambda_{x_n} R. \end{cases} \tag{I.65}$$

For TASEP and for every MC for which the uniqueness of the invariant measure is known, it suffices to find one solution: indeed, all assumptions done so far, restrict the solution space where the solution is searched, but everything is justified if the solution found satisfies all of them.

Condition (I.65) says that L and R are left and right eigenvectors of several linear combinations of A_x which is a much restrictive condition. Derrida & al. manage to find (L, A_x, R) satisfying all these constraints, and then, they characterized the invariant distribution of TASEP.

Following the idea to search necessary and/or sufficient conditions on (L, A_x, R) a more general sufficient condition on μ_n to be an invariant distribution is

$$\begin{cases} O_{c,d} & = & -B_c A_d + A_c B_d \\ L\left(\sum_a \beta^\ell[a, x_1] A_a - \beta^\ell[x_1, a] A_{x_1}\right) & = & L B_{x_1}, \\ \left(\sum_a \beta^r[a, x_n] A_a - \beta^r[x_n, a] A_{x_n}\right) R & = & -B_{x_n} R, \end{cases} \tag{I.66}$$

for some matrices $(B_x, x \in E_\kappa)$, whose sizes are compatible with the operations in which these matrices are involved (this can be checked by computing the remaining terms in the telescopic sum). When the matrices B_x are numbers, this gives the usual matrix ansatz, but when the matrices B have size > 1, this is more general since L (or R) does not need to be a common eigenvector to all the A_j's.

We may also design some variants where (L, A_{x_i}, R) would be replaced by (L_n, A_{i,x_i}, R_n) since we do not need these matrices to be the same for all n (the problem is to find the solution, and it may be easier to work with different matrices $(A_{j,x})$ instead of identical matrices (A_x), even if such a representation is possible by Lemma I.3.5.1).

One of the difficulties of this approach for the one who searches the invariant distribution of a particular Markov process on the segment, is that the "solution space" is huge, and many different solutions (L, A_x, R) representing the same measures may exist, which makes more complex the algebraic approach.

Telescopic scheme for AI MDs. More generally, the telescopic scheme occurs in the segment equilibrium equations if there exists a family of functions $\{s_i : E_\kappa^n \to \mathbb{R}, 0 \le i \le n-1\}$ such that

$$\begin{aligned} t_i(x[\![1,n]\!]) &:= \sum_{a,b} f_n(x[\![1,i-1]\!]ab\,x[\![i+2,n]\!])\mathsf{T}_{[a,b|x_i,x_{i+1}]} - f_n(x[\![1,n]\!])\mathsf{T}_{[x_i,x_{i+1}|a,b]} \\ &= -s_{i-1}(x[\![1,n]\!]) + s_i(x[\![1,n]\!]) \end{aligned}$$

for all $i \in [\![1, n-1]\!]$ and

$$y_l(x[\![1,n]\!]) := \sum_a \beta^\ell[a, x_1] f_n(a x[\![2,n]\!]) - \beta^\ell[x_1, a] f_n(x[\![1,n]\!]), \tag{I.67}$$

$$y_r(x[\![1,n]\!]) \;:=\; \sum_a \beta^r[a,x_n]f_n(x[\![1,n-1]\!]a) - \beta^r[x_n,a]f_n(x[\![1,n]\!]). \tag{I.68}$$

The functions s_i's do not need to coincide with μ_{n-1} in any respect.

Let us refocus on AI MD. Reformulate $\text{Line}_n^{M,\mathsf{T}} \equiv 0$ (given in (I.61)) for each $n \in \mathbb{N}$

$$\text{Line}_n^{M,\mathsf{T}}(x[\![1,n]\!]) = y_l^n(x[\![1,n]\!]) + \sum_{i=1}^{n-1} t_i^n(x[\![1,n]\!]) + y_r^n(x[\![1,n]\!]) \tag{I.69}$$

for $j \in [\![1,n-1]\!]$

$$y_l^n(x[\![1,n]\!]) \;:=\; \sum_{\substack{x_{-1},x_0\\x_{n+1},x_{n+2}}} \mu(x[\![-1,n+2]\!])\mathsf{Z}_{x[\![-1,2]\!]} \tag{I.70}$$

$$t_j^n(x[\![1,n]\!]) \;:=\; \sum_{\substack{x_{-1},x_0\\x_{n+1},x_{n+2}}} \mu(x[\![-1,n+2]\!])\mathsf{Z}_{x[\![j-1,j+2]\!]} \tag{I.71}$$

$$y_r^n(x[\![1,n]\!]) \;:=\; \sum_{\substack{x_{-1},x_0\\x_{n+1},x_{n+2}}} \mu(x[\![-1,n+2]\!])\mathsf{Z}_{x[\![n-1,n+2]\!]}, \tag{I.72}$$

for $\mu(x[\![-1,n+2]\!]) = \rho_{x_{-1}} \prod_{j=-1}^{n+1} M_{x_j,x_{j+1}}$. The following theorem gives a representation like this in the case of MDs: to be clear, the various parts of Theorem I.2.1.2 explain that $\text{NLine}^{M,\mathsf{T}}$ satisfies a telescopic scheme when the (ρ, M) MD is AI by T on the line. This is clearly the role of $\text{Master}_7^{M,\mathsf{T}}$ to produce the related simplifications. But the matrix ansatz is a telescopic scheme that concerns Line not NLine, so that there is a difference of nature between the matrix ansatz and the content of Theorem I.2.1.2.

The next theorem fills this gap by making the desired connection between the two approaches:

Theorem I.3.5.2
If $(L,m) = (2,1)$, $\#E_\kappa < +\infty$ and M is a MK with positive entries on E_κ. The (ρ, M) MD is AI by T on the line iff there exists a function $W : E_\kappa^3 \to \mathbb{R}$ so that for all $n \in \mathbb{N}$, $(y_l^n, y_r^n, \{s_j^n : E_\kappa^n \to \mathbb{R}, 0 \le j \le n-1\})$ defined by

$$s_j^n(x[\![1,n]\!]) \;:=\; \sum_{\substack{x_{-1},x_0\\x_{n+1},x_{n+2}}} \mu(x[\![-1,n+2]\!])\left(W(x[\![j,j+2]\!]) - W(x[\![-1,1]\!])\right), \tag{I.73}$$

$$y_l^n(x[\![1,n]\!]) \;:=\; \sum_{\substack{x_{-1},x_0\\x_{n+1},x_{n+2}}} \mu(x[\![-1,n+2]\!])(W(x[\![0,2]\!]) - W(x[\![-1,1]\!])), \tag{I.74}$$

$$y_r^n(x[\![1,n]\!]) \;:=\; -\sum_{\substack{x_{-1},x_0\\x_{n+1},x_{n+2}}} \mu(x[\![-1,n+2]\!])(W(x[\![n-1,n+1]\!]) - W(x[\![n,n+2]\!])) \tag{I.75}$$

satisfies $(y_l^n, y_r^n) = (s_0^n, -s_{n-1}^n)$ and $t_j^n(x[\![1,n]\!]) = -s_{j-1}^n(x[\![1,n]\!]) + s_j^n(x[\![1,n]\!])$, $1 \le j \le n-1$.

Remark I.3.5.3
One can find also a direct link between our approach and the matrix ansatz on the segment. Using Theorem I.2.6.3 and Theorem I.3.5.2 it can be proved that if a Markov measure is AI on the segment $[\![1,n]\!]$ (as defined in Section I.2.6) then it follows a telescopic scheme, and this for

any $n \geq 1$.

The following lemma will be a step of the proof of the theorem:

Lemma I.3.5.4
Consider $M > 0$ and $\#E_\kappa < +\infty$. If the MD (ρ, M) is AI by T on the line, then for any W satisfying Theorem I.2.1.2 (ix) there exists a constant $\alpha \in \mathbb{R}$ such that, for any $x_n, x_1 \in E_\kappa$,

$$\alpha \rho_{x_1} = \rho_{x_1} \sum_{x_{n+1}, x_{n+2}} M_{x_n, x_{n+1}} M_{x_{n+1}, x_{n+2}} W(x[\![n, n+2]\!]) \tag{I.76}$$

$$= \sum_{x_{-1}, x_0} \rho_{x_{-1}} M_{x_{-1}, x_0} M_{x_0, x_1} W(x[\![-1, 1]\!]). \tag{I.77}$$

Proof. Consider (ρ, M) MD AI by T. By Theorem I.2.1.2 (ii) and (vii), $\mathsf{NCycle}_n^{M,\mathsf{T}} \equiv 0$ and $\mathsf{NLine}_n^{M,\mathsf{T}} \equiv 0$, hence for any $x[\![1, n]\!] \in E_\kappa$

$$\rho_{x_1} \mathsf{NCycle}_n^{M,\mathsf{T}}(x[\![1, n]\!]) - \mathsf{NLine}_n^{M,\mathsf{T}}(x[\![1, n]\!]) = 0. \tag{I.78}$$

Using the expansion with the Z's, it is easily seen that all Z involved, whose parameters does not use $x_{-1}, x_0, x_{n+1}, x_{n+2}$ in $\mathsf{NLine}_n^{M,\mathsf{T}}(x[\![1, n]\!])$ are simplified by terms of $\rho_{x_1} \mathsf{NCycle}_n^{M,\mathsf{T}}(x[\![1, n]\!])$. Now fix any function W satisfying Theorem I.2.1.2 (ix). The remaining contributions in the LHS of (I.78) coming from $\rho_{x_1} \mathsf{NCycle}_n^{M,\mathsf{T}}(x[\![1, n]\!])$ are

$$\rho_{x_1}(\mathsf{Z}^{M,\mathsf{T}}_{x_{n-2}, x_{n-1}, x_n, x_1} + \mathsf{Z}^{M,\mathsf{T}}_{x_{n-1}, x_n, x_1, x_2} + \mathsf{Z}^{M,\mathsf{T}}_{x_n, x_1, x_2, x_3}) = \rho_{x_1}(W(x[\![1, 3]\!]) - W(x[\![n-2, n]\!])) \tag{I.79}$$

since $\mathsf{Z}^{M,\mathsf{T}}_{a,b,c,d} = W(b, c, d) - W(a, b, c)$. The remaining terms coming from $\mathsf{NLine}_n^{M,\mathsf{T}}(x[\![1, n]\!])$ are

$$\sum_{x_{-1}, x_0} \rho_{x_{-1}} M_{x_{-1}, x_0} M_{x_0, x_1} (\mathsf{Z}^{M,\mathsf{T}}_{x[\![-1,2]\!]} + \mathsf{Z}^{M,\mathsf{T}}_{x[\![0,3]\!]}) \tag{I.80}$$

$$+ \sum_{x_{n+1}, x_{n+2}} \rho_{x_1} M_{x_n, x_{n+1}} M_{x_{n+1}, x_{n+2}} (\mathsf{Z}^{M,\mathsf{T}}_{x[\![n-1,n+2]\!]} + \mathsf{Z}^{M,\mathsf{T}}_{x[\![n-2,n+1]\!]}) \tag{I.81}$$

$$= - \sum_{x_{-1}, x_0} \rho_{x_{-1}} M_{x_{-1}, x_0} M_{x_0, x_1} W(x[\![-1, 1]\!]) + \rho_{x_1}(W(x[\![1, 3]\!]) - W(x[\![n-2, n]\!])) \tag{I.82}$$

$$+ \sum_{x_{n+1}, x_{n+2}} \rho_{x_1} M_{x_n, x_{n+1}} M_{x_{n+1}, x_{n+2}} W(x[\![n, n+2]\!]) \tag{I.83}$$

Gathering everything, we obtain

$$\sum_{x_{n+1}, x_{n+2}} \rho_{x_1} M_{x_n, x_{n+1}} M_{x_{n+1}, x_{n+2}} W(x[\![n, n+2]\!]) - \sum_{x_{-1}, x_0} \rho_{x_{-1}} M_{x_{-1}, x_0} M_{x_0, x_1} W(x[\![-1, 1]\!]) = 0. \tag{I.84}$$

This, in turn implies that $\sum_{x_{n+1}, x_{n+2}} M_{x_n, x_{n+1}} M_{x_{n+1}, x_{n+2}} W(x[\![n, n+2]\!])$ does not depend on x_n, thus it is a constant, from what we get the result. □

Proof of Theorem I.3.5.2. Assume that (ρ, M) is AI by T on the line. Take one fixed W given by Theorem I.2.1.2 *(ix)* and define s_i^n as in (I.73) which will take the place of s_i. For $j \in [\![1, n-1]\!]$ satisfies

$$t_i^n([\![1, n]\!]) = \sum_{\substack{x_{-1}, x_0 \\ x_{n+1}, x_{n+2}}} \mu(x[\![-1, n+2]\!])(W(x[\![j, j+2]\!]) - W(x[\![j-1, j+1]\!])) \tag{I.85}$$

$$= \quad -s_{j-1}^{n}(x[\![1,n]\!]) + s_j^n(x[\![1,n]\!]). \tag{I.86}$$

In the left boundary notice that

$$y_l = \sum_{\substack{x_{-1},x_0 \\ x_{n+1},x_{n+2}}} \mu(x[\![-1,n+2]\!])(W(x[\![0,2]\!]) - W(x[\![-1,1]\!])) = s_0. \tag{I.87}$$

Recall that $\rho = \rho M$ for all $b \in E_\kappa$ and that $\sum_{b\in E_\kappa} M_{a,b} = 1$ for all $a \in E_\kappa$. Multiplying by $\prod_{k=1}^{n-1} M_{x_k,x_{k+1}}$ in Lemma I.3.5.4 and replace by $\mu(x[\![-1,n+2]\!])$ we get that

$$\sum_{\substack{x_{-1},x_0 \\ x_{n+1},x_{n+2}}} \mu(x[\![-1,n+2]\!])W_{x[\![-1,1]\!]} = \sum_{\substack{x_{-1},x_0 \\ x_{n+1},x_{n+2}}} \mu(x[\![-1,n+2]\!])W(x[\![n,n+2]\!]) \tag{I.88}$$

therefore the right boundary term satisfies that

$$y_r = -\sum_{\substack{x_{-1},x_0 \\ x_{n+1},x_{n+2}}} \mu(x[\![-1,n+2]\!])(W(x[\![n-1,n+1]\!]) - W(x[\![n,n+2]\!])) = -s_{n-1}. \tag{I.89}$$

Conversely, for n large enough and $i = \lfloor n/2 \rfloor$, for all $x[\![1,n]\!] \in E_\kappa^n$,

$$\mathsf{Z}^{M,\mathsf{T}}_{x[\![i,i+3]\!]} = \frac{t_i^n(x[\![1,n]\!])}{\mu(x[\![1,n]\!])} = \frac{s_i^n(x[\![1,n]\!]) - s_{i-1}^n(x[\![1,n]\!])}{f_n(x[\![1,n]\!])} = W(x[\![i+1,i+3]\!]) - W(x[\![i,i+2]\!]) \tag{I.90}$$

and then (ρ, M) is AI by T on the line by Theorem I.2.1.2 (ix). □

I.4 Applications

I.4.1 Explicit computation : Gröbner basis

In this subsection we generalize and revisit some well known models using our theorems. Before that, we would like to discuss a bit the "explicit" resolution of systems of algebraic equations.

First, the simplest systems of equations are linear systems: they are systems of polynomial equations of degree 1 in some unknown variables $(x_1, ..., x_n)$, with some coefficients in $\mathbb{R}$ or, possibly, with coefficients being some functions of some parameters $(y_1, \cdots, y_n)$. Such systems can be solved using linear algebra. If some parameters (y_i) are present, then the study is in general much more complicated: typically, even the dimension of the set of solutions can vary when the parameters change.

To solve these systems a computer algebra system can be used: only simple operations as multiplications, additions are needed: if the coefficients are integers, or for examples, polynomials in the y_i's with integer coefficients, the results obtained are exact.

For polynomial system with only one unknown x, of the form Sys $= \{P_i(x) = 0, 1 \leq i \leq k\}$, the first step is the computation of the $\gcd$ G of these polynomials (using Euclidean algorithm): x is solution to the system Sys if and only if $G(x) = 0$. Assuming that the P_i are not all 0 (in which case the question is trivial, but what follows does not work) if G is a constant, then Sys has no solution, and if G is a polynomial with degree bigger or equal than 1, then the solutions of Sys is the set of roots of G, which exists in $\mathbb{C}$ by d'Alembert-Gauss theorem. Finding explicit solutions can be done by numerical approximations, and in some cases, explicit exact solutions can be found; in any case, the set of solutions of Sys is implicitly known by $G(x) = 0$.

Here the situation we face is more complex: Take for example $\text{Master}_7^{M,\mathsf{T}} \equiv 0$ in the case where E_κ is finite. This system is linear in T and involves quotient of cubic monomials in the $M'_{i,j}s$. We can transform this system into a polynomial systems in several variables as follows: A pair (M,T) solves the system $S = \{\text{Cycle}_7^{M,\mathsf{T}} \equiv 0, M > 0\}$ iff it solved the following system of **polynomial** equations

$$S' := \begin{cases} \text{Cycle}_7^{M,\mathsf{T}}(x[\![1,7]\!]) &= 0 \quad ,\forall x[\![1,7]\!] \in E_\kappa^7, \\ M_{a,b}g_{a,b} - 1 &= 0 \quad ,\forall (a,b) \in E_\kappa^2 \\ -1 + \sum_{b\in E_\kappa} M_{a,b} &= 0 \quad ,\forall a \in E_k \end{cases},$$

where the $g_{a,b}$ are additional variables which prevent the $M_{a,b}$'s to be 0.

Equivalence of systems means here that (M,T) is solves S iff there exists g such that (M,T,g) solves S', and $M > 0$. Any M such that (M,T,g) solves S' has non zero entries, but could have some negative ones, or even complex ones: it depends how/where the system is solved.

We then need to solve polynomial systems in several variables. In this case again, we cannot expect a better situation than for polynomial systems of a single variable: in general no closed formulas exist for solutions, but again, it is possible to know if solutions exist, and in this case, find some minimal representations of the solution set (if T is given, the problem is almost the same).

A common way to solve this kind of problems amounts to computing a Gröbner basis: given a finite set of polynomials $S = \{P_i, i \in I\}$ where the P_i's belong to $\mathbb{R}[x_1,...,x_n]$, a Gröbner basis of S is a basis of the ideal generated by S which have some additional properties. It depends on a good monomial order (preserved by multiplications, if $x^{(\alpha)} < x^{(\beta)}$ then $x^{(\alpha)}x^{(\gamma)} < x^{(\beta)}x^{(\gamma)}$ where $x^{(\alpha)} = \prod_i x_i^{\alpha_i}$ for $\alpha = (\alpha_1,\dots,\alpha_n)$). We cannot go too far in the description of the Gröbner basis properties, or to explain how they are computed: we refer the interested reader to Adams and Loustaunau [2] to get an overview and to Jean-Charles Faugère webpage [46] for many resources on this topic, including fast algorithms.

In order to be understandable to the reader unaware of these methods, we will just stress on the following fact:

■ Computation of Gröbner basis for polynomials with integer coefficients relies on simple elementary operations as Euclidean division of polynomials, sorting of polynomials according to their coefficients/and or degrees and then can be performed by a computer algebra system working on integers (and then it is decidable).

■ When the basis B has been computed, the basis is a finite sequence of polynomials, equivalent to the initial system S.

– if the Gröbner basis is $G = [1]$ then there are no solution to the initial system (whatever is the order used),

– if it is not $G = [1]$ then there are some solutions to the initial system in $\mathbb{C}$: some extra work could be needed to see if there are some solutions in $\mathbb{R}$, $\mathbb{R}^+$ or $[0,1]^n$ if these are some additional requirements,

– since B is a basis of the ideal generated by S, each polynomial p in B is a necessary condition on the solution set. Hence if a Gröbner basis contains a polynomial, for example $(2x_1 + x_7 - 9)(3x_7 - 8x_9^{17} + 1)$ for a system $\{P_i, 1 \leq i \leq k\}$ in the variables $x_1,\cdots,x_{100}$: then they are some solutions to the system in $\mathbb{C}^{100}$, and each solution $(x_1, \cdots, x_{100})$ satisfies either $2x_1 + x_7 - 9 = 0$ or $3x_7 - 8x_9^{17} + 1 = 0$ (inclusive "or" of course).

■ Computing a Gröbner basis is time and memory consuming, so computing a Gröbner basis is sometimes impossible in practice by hand, and even by computer.

■ There are different notions of Gröbner basis as said above, since they rely on a (good) order on monomials; this order is also needed to define the Euclidean division in the set of polynomials in several variables. Each order leads to a specific representation of the ideal. For example, if $P_1 = x^2 + y^2 - z^2 - 3, P_2 = x^2 + 2y^2 - 4, P_3 = y^2 + 3z^2 - x - 2$, the computation of the Gröbner basis relative to the graded reverse lexicographical order gives as a basis $G = [2z^2 - x - 1, 2y^2 + x - 1, x^2 - x - 3]$. If alternatively, the lexicographical order (plex) is chosen, the basis is $G = [4z^4 - 6z^2 - 1, y^2 + z^2 - 1, -2z^2 + x + 1]$. Both results ensure the existence of solutions in $\mathbb{C}^3$. It is somehow trivial in this case that solution exists if we take as granted the equivalence to solve the initial system $\{P_1 = 0, P_2 = 0, P_3 = 0\}$ and (one of) the system(s) G. If we add the polynomial $P_4 = xz - y^2 + 2$, then this time (any) Gröbner basis is $G = [1]$: there are no solutions. What happens here is different from the "linear algebra settings": the number of polynomials in the Gröbner basis depends on the order chosen, and often, the basis is huge, containing many more polynomials than the initial system.

Until now, our main theorems assert that "a M-Markov law" is invariant by a PS with JRM T can be reduced to checking if a polynomial system in (M, T) has some solutions. The message here is that checking the existence of a M that solves for example $\mathsf{Cycle}_5^{M,\mathsf{T}} = 0$ when M is fixed, is possible at the price of computing a Gröbner basis. If the Gröbner basis is $G = [1]$ there are no solutions. If G is not $[1]$, it will be a list of polynomials in the variables M and T simpler than the initial problem (the order plex allows to obtain a kind of triangular system in which a well chosen order of the variables make apparent the conditions on T, for example). Again, a work still remains to be done to check that real solutions exist.

When a solution (M, T) has been found by this mean, an independent proof of the invariance of the Markov chain with kernel M by T can be done by checking directly – without using a Gröbner basis computation – that $\mathsf{Cycle}_7^{M,\mathsf{T}} \equiv 0$.

Remark I.4.1.1
There are some good reasons to be confident on the Gröbner basis computations with computer algebra systems which rely on simple computations on integers and which are used by many users, for many reasons including cryptography motivations, but for the reader which prefers to stay away from this kind of automatic tools, we insist on the fact that these computations can be done by hands (and patience).

To follow in details the following examples, the reader can download in [49] a maple-file or a pdf file, where all the computations are done.

Stochastic Ising models

The stochastic Ising model (given below Def. I.1.1.1) possesses a unique Markovian invariant measure on the line with kernel M characterized by

$$M_{0,1} = \frac{1}{1 + e^{2\beta}} \quad \text{and} \quad M_{1,1} = \frac{1}{1 + e^{-2\beta}} = M_{0,0}. \tag{I.91}$$

When $\beta = 0$, this is the Bernoulli(1/2) product measure ([84, Introduction]).

Let us see how to recover this with our approach. Since M and T are given. By Theorem I.3.1.2, since $m = 1$, $\kappa = 2$, and the range is $L = 3$, it suffices to check that Cycle$_9^{M,T} \equiv 0$. Plug the values of M and of T (given in (I.91) and in (I.5)) in the corresponding Z (which is found in (I.52)). Here Z has 5 indices, and then 32 values $Z_{a,b,c,d,e}$ need to be computed: one finds that these 32 values are all zeroes! As a consequence Cycle$_9^{M,T} \equiv 0$.

Assume now, that the existence of an invariant Markov law is unknown for this PS. Let us see how to recover this property. Again, since the range is $L = 3$, we need to find a M, for which Cycle$_9^{M,T}(a,b,c,d,e,0,0,0,0) = 0$ for all $a,b,c,d,e \in E_2$ as specified by Theorem I.3.1.2. First, we make rid of the "exponential function" in T by a change of variables and a deterministic linear change of time (the computation of a Gröbner basis must be done in a polynomial ring). To do this, we set $x = e^{-\beta}$ and use $\bar{\mathsf{T}} = x^2\mathsf{T}$ instead of T, since this does not alter the set of invariant distributions. We obtain

$$\bar{\mathsf{T}}_{[a,b,c\,|\,a,1-b,c]} = x^2 \mathsf{T}_{[a,b,c|a,1-b,c]} = x^{2+(2b-1)(2a+2c-2)}.$$

We also add the polynomials $M_{a,b}g_{a,b} - 1$ in the basis computation (this prevents each $M_{a,b}$ to be 0, as explained in section I.4.1) and for simplicity we imposed $M_{i,0} = 1 - M_{i,1}$ for all $i \in E_2$. Then with a computer algebra system, compute the Gröbner basis: this is immediate, and two solutions appear, one of them being negative. The unique positive solution (after inverting the change of variable) is given in (I.91).

The voter model and some variants

Consider the JRM T of the voter model: T is not identically 0 and besides, the voter model possesses 0^n and 1^n as absorbing states on $\mathbb{Z}/n\mathbb{Z}$ (and this can be generalized if more "opinions" are represented). The following corollary is an immediate consequence of Theorem I.2.1.5.

Corollary I.4.1.2
The only invariant Markov distributions for the voter model (or in the generalized model with κ opinions) are the mixtures on the Dirac measures on an opinion $\delta_{j^{\mathbb{Z}}}$ for $j \in E_\kappa$.

To our knowledge this theorem is so far not known. Let us now consider some variants of the voter model and the existence or not of invariant Markov law for them. Consider a JRM T in which all the entries of T are taken equal to 0 except for $\mathsf{T}_{[a,b,c|a,b',c]}$ for $b' \neq b$, $b' \in \{a,c\}$: In words, this is the voter model in which the rate at which an individual change his minds is a function of its own opinion and of those of its neighbours. A Gröbner basis for the sequence of polynomials $\{M_{a,b}g_{a,b} - 1, a,b \in \{0,1\}, \text{Cycle}_9^{M,\mathsf{T}}(a,b,c,d,e,0,0,0,0) \equiv 0\}$ gives

$$\begin{gathered}
\mathsf{T}_{[1,0,1|1,1,1]},\ \mathsf{T}_{[0,1,0|0,0,0]},\ g_{1,0}g_{1,1} - g_{1,0} - g_{1,1}, g_{0,0}g_{0,1} - g_{0,0} - g_{0,1},\\
M_{1,1}g_{1,1} - 1,\ g_{1,0}M_{1,1} - g_{1,0} + 1, M_{0,1}g_{0,1} - 1,\ M_{0,1}g_{0,0} - g_{0,0} + 1,\\
-M_{1,1}\mathsf{T}_{[0,1,1|0,0,1]}g_{0,0} - M_{1,1}\mathsf{T}_{[1,1,0|1,0,0]}g_{0,0} + \mathsf{T}_{[0,0,1|0,1,1]} + \mathsf{T}_{[1,0,0|1,1,0]}
\end{gathered}$$

Hence if a M-Markov law with positive-entries Markov kernel M is invariant then

$$\mathsf{T}_{[0,1,0|0,0,0]} = \mathsf{T}_{[1,0,1|1,1,1]} = 0 \tag{I.92}$$

(which then excludes the original voter model). From here if we replace $g_{a,b} = 1/M_{a,b}$ in the basis and look at the remaining equations, apart those corresponding to $M_{i,0} + M_{i,1} = 1$ and to the non nullity of the $M_{a,b}$'s, hen it only remains:

$$M_{0,0}(\mathsf{T}_{[0,0,1|0,1,1]} + \mathsf{T}_{[1,0,0|1,1,0]}) - M_{1,1}(\mathsf{T}_{[0,1,1|0,0,1]} + \mathsf{T}_{[1,1,0|1,0,0]}) \tag{I.93}$$

whose nullity is the only constraint (together with (I.92)) for the Markov law with positive-entries kernel M to be invariant by T on the line (since $M_{0,0}/M_{1,1}$ can take any value in $(0,+\infty)$, this is a trivial system to solve from here: there is a Markov law invariant by this dynamic on the line iff $\mathsf{T}_{[0,1,1|0,0,1]} + \mathsf{T}_{[1,1,0|1,0,0]}$ and $\mathsf{T}_{[0,0,1|0,1,1]} + \mathsf{T}_{[1,0,0|1,1,0]}$ are both positive (or both 0, in which case all Markov laws are invariant).

The contact process and some extensions

The Dirac measure on $0^{\mathbb{Z}}$ is invariant for the contact process. Another invariant distribution exists for λ large enough with no atom at $0^{\mathbb{Z}}$ (Liggett [84, Theo.1.33, Sec. VI]). We prove that this other invariant distribution is not Markovian with memory m, for any $\lambda > 0$ and any $m \geq 1$. The JRM T of the contact process is not identically 0 and the contact process possesses 0^n as absorbing state on $\mathbb{Z}/n\mathbb{Z}$, thus, an immediate consequence of Theorem I.2.1.5 is:

Corollary I.4.1.3
If a distribution $\mu \neq \delta_{0^{\mathbb{Z}}}$ is invariant for the contact process on the line, then μ is not a Markov law with memory m, for any m.

In fact, Theorem I.2.1.5 just states the non existence of invariant Markov law with memory m with positive-entries kernel. By the nature of the contact process, no other kernels are neither possible.

When solving the system for general rates and using $\mathsf{T}_{[0,0,0|0,1,0]}$ as a free parameter, meaning that it can take any real value, we found that a necessary condition to have a Markov law invariant by T is that $\mathsf{T}_{[0,0,0|0,1,0]} > 0$ which means that there is a sort of "spontaneous infection".

Around TASEP

The TASEP is the PS defined on the line, or on a segment (see Section I.2.6) whose JRM T is null, except for $\mathsf{T}_{[1,0|0,1]} = 1$. Some variants of this model have been defined, we will explore some of them.

3-colored TASEP The 3 colored model ($E_\kappa = \{0,1,2\}$) for which again the JRM is null except for

$$\mathsf{T}_{[1,0|0,1]} = \mathsf{T}_{[2,0|0,2]} = \mathsf{T}_{[2,1|1,2]} = 1 \tag{I.94}$$

meaning that a particle can overtake smaller ones, with a constant rate. For more information on this type of PS and its invariant measure on some special cases see Angel [6] or Zhong-Jun & al [35].

Here, we propose to replace the common value (I.94) by parameters, and use our theorems to characterize the set of 3-tuple $(\mathsf{T}_{[1,0|0,1]}, \mathsf{T}_{[2,0|0,2]}, \mathsf{T}_{[2,1|1,2]})$ for which an invariant Markov law with positive-entries MK M exists (T being null besides). The computation of the Gröbner basis of the system (with the additional polynomials $M_{i,j}g_{i,j} - 1$ to prevent the $M_{i,j}$ to be 0) is rapid, but the expression of a Gröbner basis is too large to be written here. What can be observed is that a polynomial of the basis is $-\mathsf{T}_{[2,0|0,2]} + \mathsf{T}_{[2,1|1,2]} + \mathsf{T}_{[1,0|0,1]}$ so that the nullity of this polynomial is a necessary condition for existence of an invariant Markov law, in which case appears that M must have constant lines, so that the distribution is a product measure with marginal $M_{0,\cdot}$. Examining further the very simple Gröbner basis, appears that any product distribution $\rho^{\mathbb{Z}}$ with ρ having support over $\{0,1,2\}$ is invariant! This can be checked by hand on $\mathsf{Cycle}_3^{\rho,\mathsf{T}} \equiv 0$.

-Variant: if particle i can overtake $i-1 \mod 3$ only. Here the parameters are

$$\mathsf{T}_{[0,2|2,0]}, \mathsf{T}_{[1,0|0,1]}, \mathsf{T}_{[2,1|1,2]},$$

meaning that now 0 can overtake 2, but not the contrary. The computation of a Gröbner basis provides a list of polynomials, among which one can find: $\mathsf{T}_{[0,2|2,0]} + \mathsf{T}_{[2,1|1,2]} + \mathsf{T}_{[1,0|0,1]}$. Since the T are non negative numbers, the 3 parameters must be 0. Hence, the only case where a Markov law with positive-entries kernel M is invariant, is when no particle are allowed to move!

-Variant with parameters $\mathsf{T}_{[a,b|b,a]}$ This is a generalization of the two previous points. In this case, each particle can overtake the other ones. This is a case where the Gröbner basis are huge (more than seven hundred polynomials), with many very simple polynomial of the following kind:

$$\mathsf{T}_{[2,0|0,2]}\mathsf{T}_{[2,1|1,2]}(g_{0,1} - g_{2,1})^2$$

meaning that one of this three factors must be 0 to have a solution. In order to study completely this system a method consists from here, to choose such an equation and to constitute 3 systems from here, each of them, constituted by the initial system at which is added one of the factor above, as a new polynomial.

Due to the complexity of the system, the constitutions of these 3-subsystems is not enough to conclude (the obtained Gröbner basis stays large), but this method can be iterated if the complete set of solutions need to be found.

Zero-range type processes We start with a preliminary definition

Definition I.4.1.4
A JRM is said to be mass preserving if $\mathsf{T}_{[a,b|c,d]} > 0 \Rightarrow a + b = c + d$.

In the literature, PS's associated to mass preserving T's are called *mass migration processes (MMP)* [45] and *mass transport models* [55, 44].

The following definition will be useful to define this type of systems.

Definition I.4.1.5
A mass preserving T is said to be zero range mass preserving if there exists a function $g : E_\kappa^2 \to \mathbb{R}$ such that

$$\mathsf{T}_{[a,b|c,d]} = g(a,k)\mathbf{1}_{(c,d)=(a-k,b+k)}, \qquad \forall a,b,c,d, 1 \le k \le a.$$

Additional note.

The qualification "mass preserving" follows the fact that such a system naturally describes some dynamics which **locally** preserve the mass, if ones identify the labels (a, b) with some mass a and b. However globally, the mass can disappear. For example, in the case in which the only positive entry is $\mathsf{T}_{[1,0|0,1]} = 1$ (TASEP), the particles move to the right. If at the beginning only a finite number of particles are present in the system, the limit process for the product topology will be the empty system.

In words: the rate at which a part k of the mass a jumps to the next vertex at its right is $g(a,k)$ (for any k legal, that is $1 \leq k \leq a$).

PS's associated to zero range mass preserving T's are called *zero range mass migration processes (MMP-ZR)* [45]. These types of processes are generalizations of TASEP, since they could be interpreted as particle systems where each site can host more than one particle and where particles in the same sites can jump at the same time (See [4, 45]).

The zero-range mass migration process is a process on E_∞ whose JRM is zero range mass preserving.

Let $\rho \in \mathcal{M}(E_\kappa)$ such that $\rho_0 > 0$. In [45, Proposition 3.10] they obtained that $\rho^{\mathbb{Z}}$ is invariant for the MMP-ZR iff $\rho_a\rho_k g(k,k) = \rho_{a+k}\rho_0 g(a+k,k) \quad \forall k \geq 1,\ \forall a \geq 1$.

Definition I.4.1.6
We say that a distribution ρ on E_κ is almost-geometrically distributed if there exists a function $\mathsf{g} : E_\kappa \to \mathbb{R}^+$ such that

$$\rho_u\rho_v = \mathsf{g}(u+v) \text{ for any } (u,v) \in E_\kappa^2. \tag{I.95}$$

The support of an almost-geometric distribution can be either finite or infinite. If the support is $\mathbb{N}$, then it is a geometric distribution (since $\rho_a\rho_b = \rho_{a+b}\rho_0$).

We make a small break in the TASEP applications to present a result that holds in a more general setting.

A family of models with an infinite number of invariant product distributions

The next theorem states that the family of mass preserving kernels having almost-geometric distributions as invariant distributions, have an infinite number of invariant distributions.

Theorem I.4.1.7
If $\rho^{\mathbb{Z}}$ is AI by a mass preserving kernel T for ρ an almost-geometric distribution such that (I.53) and (I.54) hold, then for all almost-geometric distributions ν with same support as ρ, $\nu^{\mathbb{Z}}$ is also AI by T.

Proof. We use Theorem I.3.2.2. Assume that ν is AI by T. Taking into account the discussion just above, consider $E'_\kappa = \text{Supp}(\nu)$. A necessary condition for ν to be AI by T is that (I.55) holds. Theorem I.2.2.2 says: $\rho^{\mathbb{Z}}$ is AI by T iff $\text{NCycle}_3^{\rho,\mathsf{T}}(a,b,c) = 0$ for (a,b,c) such that $\rho_a\rho_b\rho_c > 0$. Now, $\text{NCycle}_3^{\rho,\mathsf{T}}(a,b,c) =$

$$\sum_{u,v} \frac{\rho_u\rho_v}{\rho_a\rho_b} \mathsf{T}_{\substack{(u,v)\\(a,b)}} - \mathsf{T}_{\substack{(a,b)\\(u,v)}} + \sum_{u,v} \frac{\rho_u\rho_v}{\rho_b\rho_c} \mathsf{T}_{\substack{(u,v)\\(b,c)}} - \mathsf{T}_{\substack{(b,c)\\(u,v)}} + \sum_{u,v} \frac{\rho_u\rho_v}{\rho_c\rho_a} \mathsf{T}_{\substack{(u,v)\\(c,a)}} - \mathsf{T}_{\substack{(c,a)\\(u,v)}}. \tag{I.96}$$

From the definition of a mass preserving kernel, the first sum can be restricted to I_{a+b}, where for any k, I_k is the set of pairs (u,v) such that $u+v=k$; the second one can be restricted to I_{b+c}, and the last, to I_{a+c}. Now, (I.95) can be used since for all $(u,v) \in I_{a+b}$, $\rho_u\rho_v/(\rho_a\rho_b) = 1$, and one sees that (I.96) rewrites in this case $\text{NCycle}_3^{\rho,\mathsf{T}}(a,b,c) =$

$$\sum_{(u,v)\in I_{a+b}} \mathsf{T}_{\substack{(u,v)\\(a,b)}} - \mathsf{T}_{\substack{(a,b)\\(u,v)}} + \sum_{(u,v)\in I_{b+c}} \mathsf{T}_{\substack{(u,v)\\(b,c)}} - \mathsf{T}_{\substack{(b,c)\\(u,v)}} + \sum_{(u,v)\in I_{a+c}} \mathsf{T}_{\substack{(u,v)\\(c,a)}} - \mathsf{T}_{\substack{(c,a)\\(u,v)}}. \tag{I.97}$$

The steps which brings us to (I.97) is valid for any (ρ, g) satisfying (I.95) so the theorem is proved. □

Now we return to the TASEP discussion.

From [45, Proposition 3.10] and Theorem I.4.1.7, we get immediately:

Corollary I.4.1.8
Consider a zero range mass preserving T, with $g : E_\kappa^2 \to \mathbb{R}$ positive in the diagonal and $\rho \in \mathcal{M}(E_\kappa)$ with $\rho_0 > 0$ such that $\rho^{\mathbb{Z}}$ is AI by T. If for some $h : E_\kappa \to [0, \infty)$, $g(b, k) = h(b)g(k, k)$ for all $b \in E_\kappa$, then $\nu^{\mathbb{Z}}$ is also AI by T, for all almost-geometric distributions ν with same support as ρ.

PushTASEP: The PushASEP is the PS defined on $E_2^{\mathbb{Z}}$ where 0 represents an empty site and 1 an occupied site. The dynamics are described as follows: each particle tries to jump to the right at rate 1, and it actually jumps if the site is empty. Moreover, each particle jumps to the closest empty site at its left with rate 1. This type of PS has range $L = \infty$. However, each configuration can be encoded by the consecutive size of the blocks along the line, where a block is constituted with an empty site together with the set of consecutive occupied sites at its left. The dynamics of the PushASEP induces a PS on the "block size process" with range $L = 2$ and $\kappa = \infty$ (all block sizes starting by 1 are possible). For this induced PS, the product measure with marginal the geometric distribution (for any parameter in (0,1) by Theorem I.4.1.7) is AI by T. This provides a description of some invariant distributions for the PushTASEP.

I.4.2 Projection and hidden Markov chain

This part also illustrates our theorems: with Theorem I.3.1.2 one can find JRM T on $E_\kappa^{\mathbb{Z}}$ for some $\kappa \geq 3$ (with more than 3 colors) having some Markovian invariant distribution. Some of them, possess some nice projection properties: they allow to characterize some PS invariant distributions on $\{0, 1\}$ (and probably of some PS with more than 2 colors) having as invariant distribution, the distribution of some hidden Markov chain (see Cappé & al. for more information on these models).

Consider T and T′ be two JRM of two PS defined respectively $E^{\mathbb{Z}}$ and $F^{\mathbb{Z}}$, where E and F are two spaces of colors such that $\#F < \#E$. Consider π a surjective map from E on F: with each color c in F, one or several colors $\pi^{-1}(c)$ of E are associated by π (on an exclusive basis).

Definition I.4.2.1
T′ is said to be the π-projection of T if for any $a, b, c, d \in F$, any $(A, B) \in \left(\pi^{-1}(a), \pi^{-1}(b)\right)$

$$\sum_{(C,D)\in\pi^{-1}(c)\times\pi^{-1}(d)} \mathsf{T}_{[A,B|C,D]} = \mathsf{T}'_{[a,b|c,d]}. \tag{I.98}$$

In words: starting from any representative (A, B) of (a, b), the total jump rate to the representatives of (c, d) does not depend on (A, B), but only on (a, b).

Lemma I.4.2.2
Consider $\eta = (\eta_t, t \geq 0)$ with $\eta_t = (\eta_t(k), k \in \mathbb{Z})$ a well defined PS defined on $E^{\mathbb{Z}}$ with JRM T, for some finite E. Assume that T′ is the π-projection of T for a surjection $\pi : E \to F$, for some

set F. Under these hypothesis $\eta' = (\eta'_t, t \geq 0)$ defined by $\eta'_t = (\pi(\eta_t(k)), k \in \mathbb{Z})$ is a PS with JRM T′. Hence, if μ is a measure invariant by T on $E^{\mathbb{Z}}$, then $\mu \circ \pi^{-1}$ is invariant by T′ on $F^{\mathbb{Z}}$.

Proof. Since the PS under investigation is translation invariant, we focus on the finite dimensional distribution evolution at the right of zero. Consider any word $(x_1, \cdots, x_k)$ whose projection is $(\pi(x_j), 1 \leq j \leq k) = (y_1, \cdots, y_k)$. Now, consider the rate of jumps from any subword $(y_\ell, y_{\ell+1}) = (a, b)$ to $(y'_\ell, y'_{\ell+1}) = (c, d)$. By definition, $\pi(x_\ell) = y_\ell, \pi(x_{\ell+1}) = y_{\ell+1}$, and $(y_\ell, y_{\ell+1})$ jumps to $(y'_\ell, y'_{\ell+1})$ if $(x_\ell, x_{\ell+1})$ jumps to any representative (c, d) that is, if its jumps to $\pi^{-1}(c) \times \pi^{-1}(d)$. Hence, the total jump rate from (a, b) to (c, d) is given by $\sum_{(C,D)\in\pi^{-1}(c)\times\pi^{-1}(d)} \mathsf{T}[x_\ell, x_{\ell+1}][C, D]$, and this is indeed $\mathsf{T}'_{[a,b|c,d]}$ (for any value $(x_\ell, x_{\ell+1}) \in \pi^{-1}(a) \times \pi^{-1}(b)$). The statement concerning the invariance of $\mu \circ \pi^{-1}$ is direct. □

There exist in the literature several definitions for the notion of hidden Markov chains. The most classical is the following:

Definition I.4.2.3
$(Y_k, k \in \mathbb{Z})$ is said to be a hidden Markov chain taking its values in $F^{\mathbb{Z}}$, if it has the following representation:
– there exists a Markov chain $(Z_k, k \in \mathbb{Z})$ taking its values in some set $E^{\mathbb{Z}}$,
– there exists a transition kernel $K = (K(a, b))_{a\in E, b\in F}$;
such that, conditionally on $Z = (Z_k, k \in [\![a, b]\!])$, the Y_j's are independent and conditionally on Z the distribution of Y_j is given by $K(Z_j, .)$.

Hence, if $(X_k, k \in \mathbb{Z})$ is a Markov chain with state space E, and $\pi : E \to F$ is a surjection (or just a map) then since the process $(\pi(X_k), k \in \mathbb{Z})$ is a hidden Markov chain. If X has initial distribution ρ at time 0, and kernel M, then

$$\mathbb{P}(Y_j = y_j, 0 \leq j \leq n) = \sum_{\substack{(x_0,\cdots,x_n)\in E^{n+1} \\ x_j\in\pi^{-1}(y_j), 0\leq j\leq n}} \rho_{x_0} \prod_{j=1}^{n} M_{x_{j-1},x_j} \tag{I.99}$$

From there, it may be checked that a hidden Markov chain is not a Markov chain in general (with any memory), since in general, (I.99) does not factorize suitably.

Now, we state the following result:

Theorem I.4.2.4
There exist some PS on $\{0, 1\}^{\mathbb{Z}}$ which admits some hidden Markov chain (which are not Markov chains) as invariant distributions.

The proof is constructive, we will provide an example. Consider the 4-tuples $(\mathsf{T}, \mathsf{T}', M, \pi)$ as follows
– Take $\pi : E_3 \to E_2$ defined by $\pi(0) = 0, \quad \pi(1) = \pi(2) = 1$.
– Take $L = 3$ (the range), and T with entries all 0 except

$$\begin{aligned}
\mathsf{T}_{[0,0,0|0,1,0]} &= 255, \quad \mathsf{T}_{[0,0,0|0,2,0]} = 15 \\
\mathsf{T}_{[0,1,0|0,0,0]} &= \mathsf{T}_{[0,2,0|0,0,0]} = 294 \\
\mathsf{T}_{[0,1,0|0,2,0]} &= \mathsf{T}_{[0,2,0|0,1,0]} = 49
\end{aligned}$$

– The projected JRM T′ has all its entries 0 except $\mathsf{T}'_{[0,0,0|0,1,0]} = 270$, $\mathsf{T}'_{[0,1,0|0,0,0]} = 294$ (notice that the jump $(0,1,0) \to (0,2,0)$ does not project on a "true jump" since it would correspond to the jump $(0,1,0) \to (0,1,0)$).
– Take the Markov kernel

$$M = \begin{bmatrix} 7/15 & 1/3 & 1/5 \\ 1/2 & 1/6 & 1/3 \\ 1/6 & 1/2 & 1/3 \end{bmatrix}$$

and initial distribution is $r_0 = 35/89, r_1 = 29/89, r_2 = 25/89$.
If $(X_j, j \geq 0)$ is a Markov chain with kernel M and initial distribution r, then $(\pi(X_k), k \geq 0)$ is not a Markov chain since μ the projected measure satisfies $\mu([1,1,1])/\mu([1,1]) = 71/106 \neq \mu([1,1])/\mu([1]) = 53/81$, when Markovianity would imply equality of these quantities; it is a hidden Markov chain.
– It remains to say that the Markov law (ρ, M) is invariant by T. This can be proved by checking that for any $a,b,c,d,e \in E_\kappa$, $\mathsf{NCycle}_9^{M,\mathsf{T}}(a,b,c,d,e,0,0,0,0) = 0$: in fact, the corresponding function $\mathsf{Z}^{M,\mathsf{T}} \equiv 0$.

Notice that, for example, the Dirac measure $\delta_1^{\mathbb{Z}}$ is invariant on the line for this PS, and the analogous on $\mathbb{Z}/n\mathbb{Z}$, but this configuration is not attractive.

I.4.3 Exhaustive solution for the $\kappa = 2$-color case with $m = 1$ and $L = 2$

Invariant Markov laws

To find all JRM T for which exists an invariant Markov law when $\kappa = 2$ can be solved completely by computation of a Gröbner basis. Instead of writing $\mathsf{T}_{[a,b|c,d]}$, we write $t_{x,y}$ where x and y are the numbers in base 10 corresponding to ab and cd seen as a number in base 2: write for short $x = (a,b)_2$, and $y = (c,d)_2$, so that $3 = (1,1)_2$, $1 = (0,1)_2$. Hence, we have $t_{i,i} = 0$ for i from 0 to 3, and $t_{3,2} = \mathsf{T}_{[1,1|1,0]}$.

Now, set $M_{0,0} = 1 - M_{0,1}$, $M_{1,0} = 1 - M_{1,1}$ so that $M_{0,1}$ and $M_{1,1}$ are the remaining variables and now write the system which contains:
▪ $\mathsf{Cycle}_7^{M,\mathsf{T}} \equiv 0$,
▪ the equations $M_{a,b} g_{a,b} - 1$ for any $a,b \in \{0,1\}$ for additional variables $g_{0,0}, \cdots, g_{1,1}$,
▪ an additional equation $(M_{0,1} - M_{1,1})x - 1$ in order to remove the i.i.d. case (treated below), for some new variables x.
The Gröbner basis, of this system, is too long to be written here; nevertheless, here are the first polynomials of the obtained basis, which provide some necessary conditions:

$$\begin{aligned} & t_{3,0},\ t_{2,1},\ t_{1,2},\ t_{0,3}, \\ & t_{0,1}t_{1,0} + t_{0,1}t_{2,0} + t_{0,2}t_{1,0} + t_{0,2}t_{2,0} - t_{1,3}t_{3,1} - t_{1,3}t_{3,2} - t_{2,3}t_{3,1} - t_{2,3}t_{3,2}, \\ & (t_{1,0} + t_{2,0} - t_{3,1} - t_{3,2})\, {M_{1,1}}^2 + (-t_{1,0} - t_{1,3} - t_{2,0} - t_{2,3})\, M_{1,1} + t_{1,3} + t_{2,3} \end{aligned}$$

Hence, $t_{3,0} = t_{2,1} = t_{1,2} = t_{0,3} = 0$ is a necessary condition. The polynomial on the second line expresses somehow "the important condition", and the third line polynomial (and subsequent, see [49]) allow to compute the kernel M. We infer that

Corollary I.4.3.1
For $\kappa = 2$. If T is mass preserving, then there does not exist any (ρ, M)-Markov law with

$M_{0,1} \neq M_{1,1}$, and with coefficients in $(0,1)$, invariant by T.

Invariant product measure

First, we claim that,

Lemma I.4.3.2
If $\kappa = 2$, then $\rho^{\mathbb{Z}}$ is invariant by T on the line iff $\mathsf{NCycle}_2^{\rho,\mathsf{T}} \equiv 0$.

Key idea.

The proof is reminiscent of the pigeonhole principle.

Proof. By Theorem I.2.2.2 (vii), it suffices to prove that $\mathsf{NCycle}_3^{\rho,\mathsf{T}} \equiv 0 \Leftrightarrow \mathsf{NCycle}_2^{\rho,\mathsf{T}} \equiv 0$. Start by the implication: from $\mathsf{NCycle}_3^{\rho,\mathsf{T}}(a,a,a) = 0 = 3\mathsf{Z}_{a,a}^{\rho,\mathsf{T}}$ we infer that $\mathsf{Z}_{a,a}^{\rho,\mathsf{T}} = 0$ for any $a \in E_\kappa = \{0,1\}$. Now, write

$$\mathsf{NCycle}_3^{\rho,\mathsf{T}}(aab) = \mathsf{Z}_{a,a}^{\rho,\mathsf{T}} + \mathsf{Z}_{a,b}^{\rho,\mathsf{T}} + \mathsf{Z}_{b,a}^{\rho,\mathsf{T}} = 0 + \mathsf{NCycle}_2^{\rho,\mathsf{T}}(a,b), \tag{I.100}$$

so that the implication holds. For the converse, note that any words w with three letters on $E_2 = \{0,1\}$ possesses one letter repeated. Given the cyclical structure of the equation $\mathsf{NCycle}_3^{\rho,\mathsf{T}}$ (that is $\mathsf{NCycle}_3^{\rho,\mathsf{T}}(abc) = \mathsf{NCycle}_3^{\rho,\mathsf{T}}(bca)$) it suffices to prove that $\mathsf{NCycle}_3^{\rho,\mathsf{T}}(aab) = 0$ for any $a,b \in E_\kappa$. Now, from $\mathsf{NCycle}_2^{\rho,\mathsf{T}} \equiv 0$, we deduce $\mathsf{Z}_{a,a}^{\rho,\mathsf{T}} = 0$, and still from (I.100), $\mathsf{NCycle}_3^{\rho,\mathsf{T}} \equiv 0$. ☐

Now solving explicitly $\mathsf{NCycle}_2^{\rho,\mathsf{T}} \equiv 0$ using a computer algebra system is possible. The result is presented in [49]; there are 5 polynomial, including the following one (product of the 2 first lines minus the third)

$$\begin{aligned}
&(t_{1,0}t_{3,0} + t_{1,0}t_{3,1} + t_{1,0}t_{3,2} + t_{1,3}t_{3,0} + t_{2,0}t_{3,0} + t_{2,0}t_{3,1} + t_{2,0}t_{3,2} + t_{2,3}t_{3,0}) && \text{(I.101)}\\
&(t_{0,1}t_{1,3} + t_{0,1}t_{2,3} + t_{0,2}t_{1,3} + t_{0,2}t_{2,3} + t_{0,3}t_{1,0} + t_{0,3}t_{1,3} + t_{0,3}t_{2,0} + t_{0,3}t_{2,3}) && \text{(I.102)}\\
&-(t_{0,1}t_{3,0} + t_{0,1}t_{3,1} + t_{0,1}t_{3,2} + t_{0,2}t_{3,0} + t_{0,2}t_{3,1} + t_{0,2}t_{3,2} + t_{0,3}t_{3,1} + t_{0,3}t_{3,2})^2 && \text{(I.103)}
\end{aligned}$$

which is the a necessary condition on t to have a product measure as invariant distribution.

I.4.4 2D applications

The criterion provided by Theorem I.2.4.1 seems to depend on all the colorings of the neighbors of Γ_0, Γ_1 and Γ_2, which represents for this last case, as many as κ^{14} possibilities, and this for each of the κ^5 different configurations in Γ_2. So, the total number of equations seems out of reach, but in fact, again, (I.32) is decomposed on a sum of $\mathbf{Z}_{x(h\cap C)}^{h\cap C,h}$ (defined in (I.37)) so that it suffices to express these functions which intersect the domain D under inspection: the contribution of each square can be computed independently. This provides a small finite set of functions with 1, 2, 3 or 4 variables (as discussed in (I.35) and in the proof of Theorem I.2.4.1): When $E_\kappa = E_2 = \{0,1\}$, this provides a small quantity of functions, each of them being a sums of at most 2^3 elementary quantities. The corresponding set of equations can be written easily, or even automatically if needed. When T is totally specified, searching invariant distribution amounts then just to solving a polynomial system with unknown $\rho_0, \cdots, \rho_{\kappa-1}$ in the set $\{(r_0, \cdots, r_{\kappa-1}) \in [0,1]^\kappa : \sum r_i = 1\}$. Here are some cases we have investigated (we insist on the fact that all these examples have been found with very few manipulations, in a very short time):

■ If all the T's are zero except $\mathsf{T}\begin{array}{|c|c|}\hline 1\,1 & 0\,0\\ 0\,1 & 1\,0\\\hline\end{array} = a, \mathsf{T}\begin{array}{|c|c|}\hline 0\,0 & 1\,1\\ 1\,0 & 0\,1\\\hline\end{array} = 1$ (this generalizes a bit (I.29)) then the Bernoulli product measure with parameter $\rho_1 \in (0,1)$ is invariant iff $a\rho_1^2 - \rho_1^2 + 2\rho_1 - 1 = 0$ (so that for a given a, the density is $\rho_1 = 1/(\sqrt{a}+1)$). This can be checked by hand with our criterion, or just using a reversibility argument, as (I.28), for example.

■ Similarly, with the same methods, one checks that if all the T's are zero except $\mathsf{T}\begin{array}{|c|c|}\hline 1\,0 & 0\,1\\ 0\,1 & 1\,0\\\hline\end{array} = a, \mathsf{T}\begin{array}{|c|c|}\hline 0\,1 & 1\,0\\ 1\,0 & 0\,1\\\hline\end{array} = b$ then for any $(a,b) \in (0,+\infty)^2$, all product measures are invariant if $a = b$, and none otherwise (the first statement is a consequence of reversibility, but reversibility cannot be used to prove the second).

■ if all the T's are zero, except $\mathsf{T}\begin{array}{|c|c|}\hline 1\,1 & 0\,1\\ 0\,0 & 0\,1\\\hline\end{array}$, then no product measures with full support are invariant.

■ if all the T's are zero, except

$$\mathsf{T}\begin{array}{|c|c|}\hline 1\,1 & 0\,1\\ 0\,0 & 0\,1\\\hline\end{array} = a, \mathsf{T}\begin{array}{|c|c|}\hline 0\,1 & 0\,0\\ 0\,1 & 1\,1\\\hline\end{array} = b, \mathsf{T}\begin{array}{|c|c|}\hline 0\,0 & 1\,0\\ 1\,1 & 1\,0\\\hline\end{array} = c, \mathsf{T}\begin{array}{|c|c|}\hline 1\,0 & 1\,1\\ 1\,0 & 0\,0\\\hline\end{array} = d$$

then if $a = b = c = d$, all Bernoulli product measure with parameter in $(0,1)$ are invariant, otherwise, there is no invariant product measure.

■ Now, if one lets many free parameters: If all the T's are 0, except those with the form

$$\mathsf{T}\begin{array}{|c|c|}\hline a\,b & a\,1-b\\ d\,c & d\,1-c\\\hline\end{array} \text{ for } a,b,c,d \in \{0,1\}.$$

In this case, the space of parameters for which there exists invariant product measures is quite complex: see [49].

■ In the 3 colors case $E_3 = \{0,1,2\}$ with all the parameters T equal to zero, except

$$\mathsf{T}\begin{array}{|c|c|}\hline i\,i & i+1 \mod 3\; i+1 \mod 3\\ i\,i & i+1 \mod 3\; i+1 \mod 3\\\hline\end{array} = a_i.$$

The set of invariant product measures with marginal having support E_2, are the measures $\rho \in \mathcal{M}(E_2)$ satisfying $a_1\rho_1^4 - a_2\rho_2^4 = a_0\rho_0^4 - a_2\rho_2^4 = 0$ and $\rho_0, \rho_1, \rho_2 > 0$.

■ We may design similarly many JRM T preserving $P_\lambda^{\mathbb{Z}^2}$ where P_λ is the Poisson(λ) distribution, by considering mass preserving T, which moreover, preserves the Poisson distribution on a square (still using Corollary I.2.3.2). Many such dynamics exist, and this can be analyzed on the square: condition on the sum m of the 4 values around the square, and interpret the 4 (multinomial) concerned variables as the number of balls in 4 urns in which m balls labeled from 1 to m have been dropped uniformly and independently (in a larger probability space). Picking a ball at random and moving it in the next urn around the square, or shifting the urns around the square, or taking a ball and reinserting it randomly in any of the other three urns, are three examples of dynamics that preserve the multinomial distribution.

For the last example, for each positive map $m \to W_m$, the following JRM T preserves $P_\lambda^{\mathbb{Z}^2}$:

$$\mathsf{T}\begin{array}{|c|c|}\hline x_1\,x_2 & y_1\,y_2\\ x_4\,x_3 & y_4\,y_3\\\hline\end{array} = W_{\|x[\![1,4]\!]\|_1} \times \frac{x_i}{3}$$

if $y[\![1,4]\!] = x[\![1,4]\!] - e_i + e_j$ for $j \in \{1,2,3,4\} \setminus \{i\}$, where e_k is the k-th canonical vector of $\mathbb{R}^4$.

I.5 Proofs

I.5.1 Proof of Theorem I.2.1.2

Before proving Theorem I.2.1.2, we establish a Lemma used all along the proof. Recall the representation formula of $\mathsf{NLine}_n^{\rho,M,\mathsf{T}}$ (for $n \geq 3$) in terms of the functions Z given in (I.19).

Lemma I.5.1.1
Assume that M is a positive-entries Markov kernel, then for any JRM T,

$$\sum_{b,c} \mathsf{Z}_{a,b,c,d} M_{a,b} M_{b,c} M_{c,d} = 0, \quad \forall a,d \in E_\kappa.$$

Proof. Just expand Z. By definition

$$\sum_{b,c} \mathsf{Z}_{a,b,c,d} M_{a,b} M_{b,c} M_{c,d} = \sum_{u,v} \sum_{b,c} \mathsf{T}_{[u,v|b,c]} M_{a,u} M_{u,v} M_{v,d} - \sum_{b,c} \mathsf{T}^{\text{out}}_{[b,c]} M_{a,b} M_{b,c} M_{c,d}.$$

This is zero since $\sum_{b,c} \mathsf{T}_{[u,v|b,c]} = \mathsf{T}^{\text{out}}_{[u,v]}$. □

Remark I.5.1.2
The Lemma does not use the fact that the (ρ, M)-Markov law is invariant by T on the line, but just a kind of local equilibrium and the form of Z. In fact, what is true in all generality is that, for any function f taking its values in $\mathbb{R}^\star$, for $\mathsf{Z}'_{a,b,c,d} = -\mathsf{T}^{\text{out}}_{[b,c]} + \sum_{u,v} \frac{f(a,u,v,d)}{f(a,b,c,d)} \mathsf{T}_{[u,v|b,c]}$ then, one has $\sum_{b,c} \mathsf{Z}'_{a,b,c,d} f(a,b,c,d) = 0$. This is particularly true if $f(a,b,c,d) = M_{a,b} M_{b,c} M_{c,d}$ or $f(a,b,c,d) = \alpha_a M_{a,b} M_{b,c} M_{c,d} \beta_d$.

To prove Theorem I.2.1.2 we will show two cyclical implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i)$ and $(v) \Rightarrow (vi) \Rightarrow (vii) \Rightarrow (viii) \Rightarrow (ix) \Rightarrow (v)$.
• **Proof of** $(i) \Rightarrow (ii)$

Key idea.

| We compare two words that differ in one letter to obtain the equations Replace.

Observe the contribution of the term with index j, in the following equation I.19, for a j in $[\![2, n-2]\!]$ that is, far from 0 and from n

$$\mathsf{NLine}_n^{\rho,M,\mathsf{T}}(x) = \sum_{\substack{x_{-1},x_0,\\ x_{n+1},x_{n+2}}} \left(\rho_{x_{-1}} \prod_{k \in \{-1,0,n,n+1\}} M_{x_k,x_{k+1}} \right) \sum_{j=0}^{n} \mathsf{Z}^{M,\mathsf{T}}_{x[\![j-1,j+2]\!]}, \tag{I.104}$$

One sees that since for any a, $\sum_b M_{a,b} = 1$, and $\rho M = \rho$, for any $1 < j < n-1$,

$$\sum_{\substack{x_{-1},x_0,\\ x_{n+1},x_{n+2}}} \mathsf{Z}^{M,\mathsf{T}}_{x[\![j-1,j+2]\!]} \rho_{x_{-1}} \prod_{k \in \{-1,0,n,n+1\}} M_{x_k,x_{k+1}} = \rho_{x_1} \mathsf{Z}^{M,\mathsf{T}}_{x[\![j-1,j+2]\!]}. \tag{I.105}$$

Take three arbitrary words $x[\![1,n]\!]$, $y[\![1,m]\!]$ and $a[\![1,7]\!]$ with letters in E_κ, and $a'_4 \in E_\kappa$. Define

$$w = x[\![1,n]\!]\, a[\![1,4]\!]\, 000\, y[\![1,m]\!] \quad \text{and} \quad w' = x[\![1,n]\!]\, a[\![1,3]\!]\, 0000\, y[\![1,m]\!],$$

we recall that the concatenation gives for example: $a[\![2,4]\!]\,b\,y[\![3,6]\!] = a_2a_3a_4yb_3b_4b_5b_6$. Using the property (I.105) and the fact that the boundary terms are the same (those for $j \in \{0,1,n-1,n\}$), we get

$$\mathsf{NLine}_N^{\rho,M,\mathsf{T}}(w) - \mathsf{NLine}_N^{\rho,M,\mathsf{T}}(w') = \rho_{x_1}\mathsf{Replace}_7^{M,\mathsf{T}}(a_1,a_2,a_3,a_4,0,0,0;0).$$

If the (ρ,M)-Markov law is invariant by T, then $\mathsf{NLine}_N^{\rho,M,\mathsf{T}} \equiv 0$ as well as $\mathsf{NLine}_{N-1}^{\rho,M,\mathsf{T}} \equiv 0$ so that $\mathsf{Replace}_7^{M,\mathsf{T}}(a_1,a_2,a_3,a_4,0,0,0;0) = 0$.

• **Proof of** $(ii) \Rightarrow (iii)$

Key idea.

> We iterate the formula $\mathsf{Replace}_7(a,b,c,d,0,0,0;0) = 0$ to obtain that $\mathsf{Replace}_7 \equiv 0$.

For $n \geq 4$ define the map $H_n : E_\kappa^{[\![1,n]\!]} \to \mathbb{R}$ by

$$H_n(a[\![1,n]\!]) := \left(\sum_{j=1}^{n-3} \mathsf{Z}^{M,\mathsf{T}}_{a[\![j,j+3]\!]}\right) \tag{I.106}$$

$$+\left(\mathsf{Z}^{M,\mathsf{T}}_{a_{n-2},a_{n-1},a_n,0} + \mathsf{Z}^{M,\mathsf{T}}_{a_{n-1},a_n,0,0} + \mathsf{Z}^{M,\mathsf{T}}_{a_n,0,0,0}\right) - (n-4)\mathsf{Z}^{M,\mathsf{T}}_{0,0,0,0}. \tag{I.107}$$

If $\mathsf{Replace}_7^{M,\mathsf{T}}(a,b,c,d,0,0,0;0) = 0$ for all $a,b,c,d \in E_\kappa$, then for $n \geq 4$,

$$H_n(a[\![1,n]\!]) - H_{n-1}(a[\![1,n-1]\!]) = 0. \tag{I.108}$$

Indeed, since $\mathsf{Replace}_7(a[\![n-3,n]\!]\,000;0) = 0$,

$$\mathsf{Z}^{M,\mathsf{T}}_{a_{n-2},a_{n-1},a_n,0} + \mathsf{Z}^{M,\mathsf{T}}_{a_{n-1},a_n,0,0} + \mathsf{Z}^{M,\mathsf{T}}_{a_n,0,0,0} + 4\mathsf{Z}^{M,\mathsf{T}}_{0,0,0,0} = \mathsf{Z}^{M,\mathsf{T}}_{a_{n-2},a_{n-1},0,0} + \mathsf{Z}^{M,\mathsf{T}}_{a_{n-1},0,0,0} + \mathsf{Z}^{M,\mathsf{T}}_{0,0,0,0} + 4\mathsf{Z}^{M,\mathsf{T}}_{0,0,0,0}.$$

Hence by (I.108), $H_n(a[\![1,n]\!]) = H_3(a_1,a_2,a_3)$ and then, depends only on $a_1, a_{2,3}$. It follows that $H_7(a[\![1,7]\!]) = H_7(a_1,a_2,a_3,a_4',a_5,a_6,a_7)$ which implies $\mathsf{Replace}_7(a[\![1,7]\!];a_4') = 0$.

• **Proof of** $(iii) \Rightarrow (iv)$

Key idea.

> $\mathsf{Replace}_7(a,b,c,d,0,0,0;0) - \mathsf{Master}_7(a,b,c,d,0,0,0) = -\mathsf{Z}_{0,0,0,0}$ which we prove that is equal to zero.

We first prove that when (iii) holds, $\mathsf{NCycle}_4^{M,\mathsf{T}} \equiv 0$. For this just observe that $\mathsf{Replace}_7(a,b,c,d,a,b,c;0) = 0$ implies

$$\mathsf{NCycle}_4^{M,\mathsf{T}}(a,b,c,d) = \mathsf{NCycle}_4^{M,\mathsf{T}}(a,b,c,0).$$

By cyclical invariance of the map $\mathsf{NCycle}_4^{M,\mathsf{T}}$ this implies that

$$\mathsf{NCycle}_4^{M,\mathsf{T}}(a,b,c,d) = \mathsf{NCycle}_4^{M,\mathsf{T}}(0,0,0,0) = 4\mathsf{Z}^{M,\mathsf{T}}_{0,0,0,0}.$$

From that equality, multiplying both sides by $M_{a,b}M_{b,c}M_{c,d}M_{d,a}$ and summing over $a,b,c,d \in E_\kappa$ we obtain

$$\begin{aligned}4\mathsf{Z}^{M,\mathsf{T}}_{0,0,0,0} \times \mathrm{Trace}(M^4) &= \sum_{a,b,c,d\in E_\kappa} \mathsf{NCycle}_4(a,b,c,d)M_{a,b}M_{b,c}M_{c,d}M_{d,a} \\ &= 4\sum_{a,d\in E_\kappa} M_{d,a}\Big(\sum_{b,c}\mathsf{Z}^{M,\mathsf{T}}_{a,b,c,d}M_{a,b}M_{b,c}M_{c,d}\Big),\end{aligned}$$

By Lemma I.5.1.1, this last quantity is 0, and then $\mathsf{Z}^{M,\mathsf{T}}_{0,0,0,0} = 0$.

To end the proof, it suffices to observe that when $\mathsf{Z}^{M,\mathsf{T}}_{0,0,0,0} = 0$, for any $a, b, c, d \in E_\kappa$,

$\mathsf{Replace}^{M,\mathsf{T}}_7(a, b, c, d, 0, 0, 0; 0) = \mathsf{Master}^{M,\mathsf{T}}_7(a, b, c, d, 0, 0, 0)$.

- **Proof of** $(iv) \Rightarrow (v)$

Key idea.

| We iterate the formula $\mathsf{Master}_7(a, b, c, d, 0, 0, 0) = 0$ to obtain that $\mathsf{Master}_7 \equiv 0$.

Suppose (iv). $\mathsf{Master}_7(0, 0, 0, 0, 0, 0, 0) = 0$ implies that $\mathsf{Z}^{M,\mathsf{T}}_{0,0,0,0} = 0$. Recall the map H_n defined in (I.106) and replace $\mathsf{Z}^{M,\mathsf{T}}_{0,0,0,0}$ by 0 inside. Using now that $\mathsf{Master}_7(a[\![n-3, n]\!]\,000) = 0$, one has for any $n \geq 4$,

$$H_n(a[\![1, n]\!]) = H_{n-1}(a[\![1, n-1]\!]).$$

Hence $H_n(a[\![1, n]\!])$ is a function of (a_1, a_2, a_3) only and it does not depend on n. Therefore, since $\mathsf{Master}_7(a, b, c, d, e, f, g) = H_7(a, b, c, d, e, f, g) - H_6(a, b, c, e, f, g)$ we see that this quantity is 0.
Proof of $(v) \Rightarrow (i)$

Key idea.

| We prove that $\mathsf{Master}_7 \equiv 0$ implies $\mathsf{NLine}_i \equiv 0$ for $i \in \{1, 2, 3\}$ and therefore for all $i \in \mathbb{N}$.

We will need three intermediate results:

Lemma I.5.1.3
Assume that M is a Markov kernel with positive entries. If $\mathsf{Master}^{M,\mathsf{T}}_7 \equiv 0$ then for all $n \geq 3$, all $m \in [\![2, n-1]\!]$,

$$\mathsf{NLine}^{\rho,M,\mathsf{T}}_n(x[\![1, n]\!]) = \mathsf{NLine}^{\rho,M,\mathsf{T}}_{n-1}(x[\![1, n]\!]^{\{m\}}), \text{ for all } x[\![1, n]\!] \in E^n_\kappa$$

and then
$$\mathsf{NLine}^{\rho,M,\mathsf{T}}_n(x[\![1, n]\!]) = \mathsf{NLine}^{\rho,M,\mathsf{T}}_2(x_1, x_n), \text{ for all } x[\![1, n]\!] \in E^n_\kappa.$$

Proof. The second statement is a consequence of the first one. Following Remark I.2.1.7, when $\mathsf{Master}^{M,\mathsf{T}}_7 \equiv 0$, we may replace any sum of the form

$$S(x[\![-1, n+2]\!]) := \sum_{j=0}^{n} \mathsf{Z}_{x[\![j-1,j+2]\!]},$$

which depends on the word $x[\![-1, n+2]\!]$, with the sum $S(x[\![-1, n+2]\!]^{\{m\}})$, that is corresponding with the same word with the letter with index m removed for any $m \in [\![2, n-1]\!]$. Hence from (I.19),

$$\begin{aligned}
\mathsf{NLine}^{\rho,M,\mathsf{T}}_n(x[\![1, n]\!]) &= \sum_{\substack{x_{-1},x_0,\\ x_{n+1},x_{n+2}}} S(x[\![-1, n+2]\!])\rho_{x_{-1}} \prod_{k\in\{-1,0,n,n+1\}} M_{x_k,x_{k+1}} \\
&= \sum_{\substack{x_{-1},x_0,\\ x_{n+1},x_{n+2}}} S(x[\![-1, n+2]\!]^{\{m\}})\rho_{x_{-1}} \prod_{k\in\{-1,0,n,n+1\}} M_{x_k,x_{k+1}}
\end{aligned}$$

and this is $\mathsf{NLine}^{\rho,M,\mathsf{T}}_n(x[\![1, n]\!]^{\{m\}})$ since m is not equal to 1 or to n. □

Lemma I.5.1.4
Assume that M is a Markov kernel with positive entries. For all $a \in E_\kappa$,

$$\sum_b \mathsf{NLine}_2^{\rho,M,\mathsf{T}}(a,b)M_{a,b} = \mathsf{NLine}_1^{\rho,M,\mathsf{T}}(a) = \sum_b \mathsf{NLine}_2^{\rho,M,\mathsf{T}}(ba)M_{b,a}.$$

and more generally, for any $x[\![1,n]\!]$ for $n \geq 1$,

$$\sum_b \mathsf{NLine}_{n+1}^{\rho,M,\mathsf{T}}(x[\![1,n]\!]b)M_{x_n,b} = \mathsf{NLine}_n^{\rho,M,\mathsf{T}}(x[\![1,n]\!]) = \sum_b \mathsf{NLine}_{n+1}^{\rho,M,\mathsf{T}}(bx[\![1,n]\!])M_{b,x_1}.$$

Proof. By (I.19) and (I.18)

$$\sum_{x_2} \mathsf{NLine}_2^{\rho,M,\mathsf{T}}(x_1,x_2)M_{x_1,x_2} = \sum_{\substack{x_{-1},x_0,x_2,\\ x_3,x_4}} \left(\rho_{x_{-1}} \prod_{k=-1}^{3} M_{x_k,x_{k+1}}\right) \sum_{j=0}^{2} \mathsf{Z}^{M,\mathsf{T}}_{x[\![j-1,j+2]\!]}.$$

The contribution of the term $j = 2$, is

$$\sum_{x_{-1},x_0} \left(\rho_{x_{-1}} \prod_{k=-1}^{0} M_{x_k,x_{k+1}}\right) \sum_{x_2,x_3,x_4} M_{x_1,x_2}M_{x_2,x_3}M_{x_3,x_4}\mathsf{Z}^{M,\mathsf{T}}_{x[\![1,4]\!]},$$

which is 0 by the Lemma I.5.1.1. Therefore

$$\sum_{x_2} \mathsf{NLine}_2^{\rho,M,\mathsf{T}}(x_1,x_2)M_{x_1,x_2} = \sum_{\substack{x_{-1},x_0,x_2,\\ x_3,x_4}} \left(\rho_{x_{-1}} \prod_{k=-1}^{3} M_{x_k,x_{k+1}}\right) \sum_{j=0}^{1} \mathsf{Z}^{M,\mathsf{T}}_{x[\![j-1,j+2]\!]};$$

the sum on x_4 simplifies (because $\sum_{x_4} M_{x_3,x_4} = 1$), which gives the expected result. The proof of the second statement and of the generalization to larger words, can be obtained similarly. □

Lemma I.5.1.5
If M is a Markov kernel with positive entries and if $\mathsf{Master}_7^{M,\mathsf{T}} \equiv 0$, then $\mathsf{NLine}_2^{\rho,M,\mathsf{T}} \equiv 0$ and $\mathsf{NLine}_1^{\rho,M,\mathsf{T}} \equiv 0$.

Proof. Using Lemma I.5.1.4 and then Lemma I.5.1.3 which asserts that one can suppress the middle letter in the argument of $\mathsf{NLine}_3^{\rho,M,\mathsf{T}}$, for any $a,b \in E_\kappa$,

$$\mathsf{NLine}_2^{\rho,M,\mathsf{T}}(ab) = \sum_c \mathsf{NLine}_3^{\rho,M,\mathsf{T}}(abc)M_{b,c} = \sum_c \mathsf{NLine}_2^{\rho,M,\mathsf{T}}(ac)M_{b,c}. \tag{I.109}$$

Set M^t the matrix transposed of M, and, for a fixed $a \in E_\kappa$, consider the row vector

$$v_a = \left[\mathsf{NLine}_2^{\rho,M,\mathsf{T}}(ab), b \in E_\kappa\right].$$

The equality between the leftmost and rightmost quantities in (I.109) can be written $v_a = v_a M^t$, so that it is apparent that v_a is a left eigenvector of M^t, associated with the eigenvalue 1. Since M is a Markov kernel with positive entries, $v_a = \lambda_a [1,\cdots,1]$ for some $\lambda_a \in \mathbb{R}$. Therefore $\mathsf{NLine}_2^{\rho,M,\mathsf{T}}(a,b) = \lambda_a$, then $\mathsf{NLine}_2^{\rho,M,\mathsf{T}}(a,b)$ does not depend on b. Now, notice that by translation invariance $\sum_a \mathsf{Line}_2^{\rho,M,\mathsf{T}}(a,b) = \sum_a \mathsf{Line}_2^{\rho,M,\mathsf{T}}(b,a)$ since both measure the balance of the state b. Since $\mathsf{Line}_2^{\rho,M,\mathsf{T}}(a,b) = \mathsf{NLine}_{a,b}^{\rho,M,\mathsf{T}}M_{a,b}$ the previous considerations lead to

$$\sum_a \lambda_a M_{a,b} = \sum_a \lambda_b M_{b,a}$$

and since the RHS is λ_b, this says $\lambda = (\lambda_a, a \in E_\kappa)$ is a right eigenvector of M associated with the eigenvalue 1, so that $\lambda_a = \alpha\rho_a$ for a constant α. It remains to compute α.

Write

$$\sum_a \mathsf{NLine}_1^{\rho,M,\mathsf{T}}(a) = \sum_a \sum_b \mathsf{NLine}_2^{\rho,M,\mathsf{T}}(a,b)M_{a,b} = \alpha \sum_a \rho_a = \alpha.$$

Now, using that $\mathsf{Master}_7^{M,\mathsf{T}} \equiv 0$, let us prove that $\alpha = 0$. For this consider

$$\begin{aligned}\alpha = \sum_{x_1} \mathsf{NLine}_1^{\rho,M,\mathsf{T}}(x_1) &= \sum_{x_{-1},x_0,x_1,x_2} \rho_{x_{-1}} \mathsf{Z}_{x_{-1},x_0,x_1,x_2} \prod_{k\in\{-1,0,1\}} M_{x_k,x_{k+1}} \\ &+ \sum_{x_0,x_1,x_2,x_3} \rho_{x_0} \mathsf{Z}_{x_0,x_1,x_2,x_3} \prod_{k\in\{0,1,2\}} M_{x_k,x_{k+1}}\end{aligned}$$

and this is 0 by Lemma I.5.1.1. Hence $\alpha = 0$, and therefore $\mathsf{NLine}_1^{\rho,M,\mathsf{T}} \equiv 0$ and $\mathsf{NLine}_2^{\rho,M,\mathsf{T}} \equiv 0$.□

Putting together the three previous Lemmas, we see that when $\mathsf{Master}_7^{M,\mathsf{T}} \equiv 0$, then $\mathsf{NLine}^{\rho,M,\mathsf{T}} \equiv 0$ for any n, and then $\mathsf{Line}^{\rho,M,\mathsf{T}} \equiv 0$ too.

- **Proof of** $(v) \Rightarrow (vi)$

Key idea.

> We use that once NCycle_n does not depend on one letter, by cyclic invariance, it does not depend on any letter.
> **Tools:** invariants.

Observe the linear form of $\mathsf{NCycle}_n^{M,\mathsf{T}}$ given in (I.23) (valid for $n \geq 3$) and check that for $n \geq 7$ and any $x \in E_\kappa^{[\![0,n-1]\!]}$,

$$\mathsf{NCycle}_n^{M,\mathsf{T}}(x) = \mathsf{NCycle}_{n-1}^{M,\mathsf{T}}(x^{\{n-4\}}) + \mathsf{Master}_7^{M,\mathsf{T}}(x[\![n-7, n-1]\!]), \tag{I.110}$$

which rewrites $\mathsf{NCycle}_n^{M,\mathsf{T}}(x) = \mathsf{NCycle}_{n-1}^{M,\mathsf{T}}(x^{\{n-4\}})$, since $\mathsf{Master}_7^{M,\mathsf{T}} \equiv 0$. This formula implies that $\mathsf{NCycle}_n^{M,\mathsf{T}}(x)$ does not depend on x_{n-4}, and then since $\mathsf{NCycle}_n^{M,\mathsf{T}}$ is cyclically invariant, does not depend on any letter. It is then equal to $\mathsf{NCycle}_n^{M,\mathsf{T}}(0^n)$ where 0^n is the word formed with n repetitions of 0, and then since $\mathsf{NCycle}_n^{M,\mathsf{T}}(0^n) = n\mathsf{Z}_{0,0,0,0}$, we can conclude using $\mathsf{Master}_7^{M,\mathsf{T}}(0^7) = Z_{0,0,0,0} = 0$.

It remains to treat the case $n = 3$ to 6. But observe (I.23), and consider for $m \in [\![3,6]\!]$ a word w of size m, and the word w^g obtained by the concatenation of g copies of w for the $g \geq 3$ of your choice. Then one sees that

$$\mathsf{NCycle}_m^{M,\mathsf{T}}(w) = \mathsf{NCycle}_{gm}^{M,\mathsf{T}}(w^g)/g = 0,$$

since $gm \geq 9$.
- **Proof of** $(vi) \Rightarrow (vii) \Rightarrow (viii)$ Trivial
- **Proof of** $(viii) \Rightarrow (ix)$. Consider the map

$$\mathsf{W}_{a,b,c} := \mathsf{Z}_{0,0,0,a} + \mathsf{Z}_{0,0,a,b} + \mathsf{Z}_{0,a,b,c}.$$

We will use $\mathsf{NCycle}_7^{M,T}(a,b,c,d,0,0,0) \equiv 0$ with different parameters. Start with $a = b = c = d = 0$ to obtain that $\mathsf{Z}_{0,0,0,0}^{M,\mathsf{T}} = 0$. Now use arbitrary $a, b, c \in E_\kappa$ and $d = 0$ to obtain that

$$\mathsf{W}_{a,b,c} = \mathsf{Z}_{0,0,0,a} + \mathsf{Z}_{0,0,a,b} + \mathsf{Z}_{0,a,b,c} = -\mathsf{Z}_{a,b,c,0} - \mathsf{Z}_{a,b,0,0} - \mathsf{Z}_{a,0,0,0}.$$

Now for arbitrary $a, b, c, d \in E_\kappa$, $\mathsf{NCycle}_7^{M,T}(a, b, c, d, 0, 0, 0) \equiv 0$ is equivalent to

$$\begin{aligned} \mathsf{Z}_{a,b,c,d} &= -\mathsf{Z}_{0,0,0,a} - \mathsf{Z}_{0,0,a,b} - \mathsf{Z}_{0,a,b,c} - \mathsf{Z}_{b,c,d,0} - \mathsf{Z}_{c,d,0} - \mathsf{Z}_{d,0,0,0} \\ &= -\mathsf{W}_{a,b,c} + \mathsf{W}_{b,c,d}. \end{aligned}$$

• **Proof of** $(ix) \Rightarrow (v)$. If $\mathsf{Z}_{a,b,c,d} = -\mathsf{W}_{a,b,c} + \mathsf{W}_{b,c,d}$, then a telescopic simplification allows us to see that for all $n \geq 4$

$$\sum_{w \in \mathsf{Seq}_4(a[\![1,n]\!])} \mathsf{Z}_w = \mathsf{W}_{a_{n-2},a_{n-1},a_n} - \mathsf{W}_{a_1,a_2,a_3}$$

from what we infer by (I.21) that $\mathsf{Master}_7^{M,\mathsf{T}} \equiv 0$. □

I.5.2 Proof of Theorem I.2.2.2

The proof is almost the same as that of Theorem I.2.1.2. The only differences concern $(v) \Rightarrow (vi)$ and $(viii) \Rightarrow (ix)$.
To prove $(v) \Rightarrow (vi)$, take the proof of the corresponding statement in Theorem I.2.1.2 noticing that now the linear form (I.27) of $\mathsf{NCycle}_n^{\rho,\mathsf{T}}$ is valid from $n \geq 2$ and replace (I.110) by

$$\mathsf{NCycle}_n^{\rho,\mathsf{T}}(x) = \mathsf{NCycle}_{n-1}^{\rho,\mathsf{T}}(x^{\{n-2\}}) + \mathsf{Master}_3^{\rho,\mathsf{T}}(x[\![n-3, n-1]\!]).$$

For $(viii) \Rightarrow (ix)$: Take $ab0 = 000$ to deduce $\mathsf{Z}_{0,0}^{\rho,\mathsf{T}} = 0$. Use this and take $ab0 = a00$ to find that $\mathsf{Z}_{a,0}^{\rho,\mathsf{T}} + \mathsf{Z}_{0,a}^{\rho,\mathsf{T}} = 0$. For general $ab0$ we obtain the identity $\mathsf{Z}_{ab}^{\rho,\mathsf{T}} = \mathsf{Z}_{0,b}^{\rho,\mathsf{T}} - \mathsf{Z}_{0,a}^{\rho,\mathsf{T}}$. So it is enough to define $\mathsf{W}_a = \mathsf{Z}_{0,a}^{\rho,\mathsf{T}} + C$ (for any constant C).

I.5.3 Proof of Theorem I.2.2.6

We will adapt the proof of Theorem 3.1. in [45] (steps 4 and 5). The main difference is that they use that a measure is invariant iff $\int Gf(\eta)d\rho^{\mathbb{Z}}(\eta) = 0$ for every bounded cylinder function $f : E_\kappa^{\mathbb{Z}^d} \to \mathbb{R}$. This is equivalent to $\mathsf{Line}^{\rho,\mathsf{T},p}(x(A)) = 0$ for any $A \subset \mathbb{Z}^d$ finite, and $x(A) \in E_\kappa^A$. We do not need to take the limit to get (91) and (92) and the last part of step 5, just n sufficiently large, given that our p is a finite rate transition probability.

I.5.4 Proof of Theorem I.2.5.1

• Let us first assume that $\nu_{a,b,c} = \frac{M_{a,b}M_{b,c}M_{c,a}}{t^3}$ for $t = \mathsf{Trace}(M^3)^{1/3}$ for some Markov kernel M. In this case, N_a given in (I.44) satisfies $N_a = (1/t)\left[\frac{M_{x,y}M_{y,a}}{M_{x,a}}\right]_{x,y \in E_\kappa}$, and then a main right eigenvector of N_a is given by $r'_a = {}^t\left[1/M_{y,a}\right]_{y \in E_k}$. One sees that $\lambda = 1/t$ is the common main eigenvalue to all the N_a's. In the same way, one sees that $\ell'_a = \left[\rho_x M_{x,a}\right]_{x \in E_k}$ is a main left eigenvector associated to N_a.

The vectors r_a and ℓ_a of the theorem are obtained after normalization: $\ell_a = \left[\rho_x M_{x,a}/\rho_a\right]_{x \in E_\kappa}$, $r_a = \rho_a r'_a = {}^t\left[\rho_a/M_{y,a}\right]_{y \in E_k}$. Now $L_{a,b} = \left(M_{a,b}M_{b,x}M_{x,a}/t^3, x \in E_\kappa\right)$, giving $L_{a,b}r_a = \rho_a M_{a,b}/t^3$

and indeed, $\sum_{a,b} L_{a,b} r_a = 1/t^3 = \lambda^3$.

• Now, assume that ν is given, and (I.42) possesses a positive recurrent solution M. From the previous point, the N_a's have same main eigenvalues. The main argument of the proof we will develop relies on the structure of ν, which allows to show that M exists, it is characterized by (I.42). Equation (I.42) motivates to consider $\nu_{a,b,c}$ as the weight of a cycle abc of length 3, which may be expanded as a product on its edges:

$$\nu_{a,b,c} = \prod_{e\in\{(a,b),(b,c),(c,a)\}} w_e,$$

where

$$w_{(u,v)} = M_{u,v}/t, \qquad \text{for } t = \text{Trace}(M^3)^{1/3}.$$

More generally, for any directed graph $G = (V, E)$, let

$$W(G) = \prod_{e\in E} w_e.$$

We will see that the knowledge of $\nu_{a,b,c}$ allows to determine the weight of the cycles of every size, and then, by taking a limit, we will determine M. First, taking $(a, b, c) = (a, a, a)$ provides

$$M_{a,a} = t\,\nu_{a,a,a}^{1/3}$$

and for the cycle (a, b, a), since $\nu_{a,b,a} = M_{a,b}M_{b,a}M_{a,a}/t^3$,

$$M_{a,b}M_{b,a} = t^2\,\nu_{a,b,a}\,\nu_{a,a,a}^{-1/3}.$$

Consider a cycle $\mathcal{C}_n = (a_1, \ldots, a_{n-1}, a_n, a_1)$ of length n on E_κ, and for some $1 < j < n$ add the directed edge (a_1, a_j) as well as the edge (a_j, a_1) to get the graph $\mathcal{C}'_n$. We may partition this oriented graph also as the union of $\mathcal{C}_j$ of length j and $\mathcal{C}_{j,n} = (a_j, \ldots, a_{n-1}, a_n, a_1, a_j)$. Therefore

$$W(\mathcal{C}_n) = \frac{W(\mathcal{C}'_n)}{w_{a_1,a_j}w_{a_j,a_1}} = \frac{W(\mathcal{C}_j)W(\mathcal{C}_{j,n})}{w_{a_1,a_j}w_{a_j,a_1}} = W(\mathcal{C}_j)W(\mathcal{C}_{j,n}) \times \frac{\nu_{a_1,a_1,a_1}^{1/3}}{\nu_{a_1,a_j,a_1}}. \tag{I.111}$$

A simple iteration argument allows one to express the weight of a cycle of any length with the weights of cycles of length 3. A particular way to do that, is to see (I.111) as the algebraic effect of the addition of the edge (a_1, a_j) and (a_j, a_1) in the cycle $\mathcal{C}_n$: adding all the edges from and to a_1 yields to

$$W(\mathcal{C}_n) = \nu_{a_1,a_2,a_3} \prod_{j=3}^{n-1} \frac{\nu_{a_1,a_j,a_{j+1}}\nu_{a_1,a_1,a_1}^{1/3}}{\nu_{a_1,a_j,a_1}}. \tag{I.112}$$

Using the matrices L, N, R, (I.112) implies that

$$\sum_{a_3,\cdots,a_n} W(\mathcal{C}_n) = L_{a_1,a_2} N_{a_1}^{n-3}\mathbf{1}.$$

Using Perron-Frobeniüs theorem and (i) (here is used the fact that the N_a's have the same eigenvalues),

$$\sum_{a_3,\cdots,a_n} \frac{W(\mathcal{C}_n)}{\lambda_{a_1}^{n-3}} = \sum_{a_3,\cdots,a_n} \frac{W(\mathcal{C}_n)}{\lambda^{n-3}} \underset{n\to+\infty}{\longrightarrow} L_{a_1,a_2} r_{a_1} \ell_{a_1} R = L_{a_1,a_2} r_{a_1} > 0. \tag{I.113}$$

It is important to notice that the formula hence obtained, is independent from the Markov kernel M solution of (I.42) chosen, so that every M which solves (I.42) must satisfy $t^n\, W(\mathcal{C}_n) = M_{a_n,a_1} \prod_{j=1}^{n-1} M_{a_j,a_{j+1}}$. Summing the previous relation over all the values of $a_3, \cdots, a_n$, we get that it must also satisfy

$$\sum_{a_3,\cdots,a_n} t^n\, W(\mathcal{C}_n) = M_{a_1,a_2} M^{n-1}_{a_2,a_1}. \tag{I.114}$$

Now we make some connections. Compare (I.113) with (I.114). Taking $\lambda = 1/t$, we see that

$$M_{a_1,a_2} M^{n-1}_{a_2,a_1} \xrightarrow[n\to+\infty]{} t^3 L_{a_1,a_2} r_{a_1}.$$

Since M is assumed to be positive recurrent, $M^{n-1}_{a_2,a_1} \to \rho_{a_1}$ for some probability measure ρ. Hence we have established that for all pair (ρ, M) where M is a positive recurrent Markov kernel M, and ρ its invariant probability measure, satisfies $\rho_{a_1} M_{a_1,a_2} = t^3 L_{a_1,a_2} r_{a_1}$ for any $(a_1, a_2) \in E^2_\kappa$. But a unique pair (ρ, M) is solution of this equation when the RHS is given, since ρ_a must be equal to $\sum_b t^3 L_{a,b} r_a$.

Appendix

Contents

I.A Proof of Theorem I.2.1.8

For any $I \subset \mathbb{N}$, denote by $\mathcal{E}_I = \{(M, \mathsf{T}) : \mathsf{NCycle}_i^{M,\mathsf{T}} \equiv 0, i \in I\}$. First, we claim that

Lemma I.A.0.1
$\mathcal{E}_{4,5,6} = \mathcal{E}_7$.

Proof. • From Theorem I.2.1.2, if $(M, \mathsf{T}) \in \mathcal{E}_7$ then $(M, \mathsf{T}) \in \mathcal{E}_{4,5,6}$.

• For the converse: we use the following identity

$$\begin{aligned}\mathsf{NCycle}_7(abcdefg) \quad = \quad & -\mathsf{NCycle}_6(efgdef) + \mathsf{NCycle}_4(efgd) + \mathsf{NCycle}_6(efadef) \\ & -\mathsf{NCycle}_4(efad) + \mathsf{NCycle}_6(efgcef) - \mathsf{NCycle}_4(efgc) \\ & -\mathsf{NCycle}_6(efacef) + \mathsf{NCycle}_4(efac) + \mathsf{NCycle}_6(abcdef) \\ & +\mathsf{NCycle}_6(abcefg) - \mathsf{NCycle}_5(abcef),\end{aligned}$$

which can be checked by expansion in terms of Z_w, and by making an inventory of the multiplicity of each word w involved. Hence, $\mathsf{NCycle}_7^{M,\mathsf{T}}$ is a linear combination of some instances of NCycle_j for j from 4 to 6. □

Here is a short explanation of the origin of the formula appearing in the lemma proof: Take (M, T) a solution of $\mathcal{E}_{4,5,6}$. Since $0 = \mathsf{NCycle}_4^{M,\mathsf{T}}(a, b, a, b)$, therefore $\mathsf{Z}_{a,b,a,b} = -\mathsf{Z}_{b,a,b,a}$. Now, since

$$\begin{aligned}0 \quad &= \quad \mathsf{NCycle}_6^{M,\mathsf{T}}(a, b, c, d, a, b) - \mathsf{NCycle}_4^{M,\mathsf{T}}(a, b, c, d) \\ &= \quad -\mathsf{Z}_{d,a,b,c} + \mathsf{Z}_{d,a,b,a} + \mathsf{Z}_{a,b,a,b} + \mathsf{Z}_{b,a,b,c},\end{aligned}$$

replacing $\mathsf{Z}_{a,b,a,b}$ by $-\mathsf{Z}_{b,a,b,a}$ in this equation, gives the identity

$$\mathsf{Z}_{d,a,b,c} - \mathsf{Z}_{d,a,b,a} - \mathsf{Z}_{b,a,b,c} + \mathsf{Z}_{b,a,b,a} \quad \forall a, b, c, d \in \mathbb{N}.$$

In these equations, the parameters of Z have the form $Z_{x,a,b,y}$ for different values of x and y. Two terms depend on d and two on c: this implies that the differences between the elements that depend on d (respectively c) do not depend on d (respectively c). This provides new identities. Playing with the dependence of the differences in the variables involve, leads eventually to the formula. But the formula can be checked directly independently from these considerations as indicated in the Theorem proof.

Proof of Theorem I.2.1.8. (ii) We start by proving that if $(M, \mathsf{T}) \in \mathcal{E}_{4,6}$, then $(M, \mathsf{T}) \in \mathcal{E}_5$. For this just check that

$$\begin{aligned}\mathsf{NCycle}_5(abcde) &= \mathsf{NCycle}_6(deecde) - \mathsf{NCycle}_6(deacde) + \mathsf{NCycle}_4(deac)\\ &\quad \mathsf{NCycle}_6(dea0de) - \mathsf{NCycle}_4(deec) - \mathsf{NCycle}_4(dea0)\\ &\quad -\mathsf{NCycle}_6(dee0de) + \mathsf{NCycle}_4(dee0).\end{aligned}$$

(i) We prove that if $(M, \mathsf{T}) \in \mathcal{E}_{5,6}$ then it is in $\mathcal{E}_4$ too. By expansion, one checks that

$$\begin{aligned}\mathsf{NCycle}_4^{M,\mathsf{T}}(a,b,c,d) &= \mathsf{NCycle}_5^{M,\mathsf{T}}(a,b,c,d,a) + \mathsf{NCycle}_5^{M,\mathsf{T}}(a,b,c,c,d)\\ &\quad -\mathsf{NCycle}_6^{M,\mathsf{T}}(a,b,c,c,d,a).\end{aligned}$$

I.B Proof of Theorem I.2.5.3

Here is a Lemma which implies Theorem I.2.5.3.

Lemma I.B.0.1
Let M and M' be two positive Markov kernels on E_κ, for $1 \le \kappa \le +\infty$. The following three properties (1) (2) and (3) are equivalent

(1) $\dfrac{M_{a,u}M_{u,v}M_{v,d}}{M_{a,b}M_{b,c}M_{c,d}} = \dfrac{M'_{a,u}M'_{u,v}M'_{v,d}}{M'_{a,b}M'_{b,c}M'_{c,d}} \quad \forall a,b,c,d,u,v.$

(2) $\dfrac{M_{a,u}M_{u,v}M_{v,a}}{M_{a,b}M_{b,c}M_{c,a}} = \dfrac{M'_{a,u}M'_{u,v}M'_{v,a}}{M'_{a,b}M'_{b,c}M'_{c,a}} \quad \forall a,b,c,u,v.$

(3) $\dfrac{M'_{a,b}M'_{b,c}}{M_{a,b}M_{b,c}} = \alpha\dfrac{M'_{a,c}}{M_{a,c}} \quad \forall a,b,c,$ for some $\alpha > 0$, independent of a,b,c.

Moreover, if M and M' are positive recurrent, then each of the previous properties implies that $M = M'$.

Proof. Taking $d = a$ in (1) suffices to see that $(1) \Rightarrow (2)$.
Proof of $(2) \Rightarrow (3)$. Taking $u = a$ and $v = c$ in (2) provides

$$\frac{M'_{a,a}M'_{a,c}}{M_{a,a}M_{a,c}} = \frac{M'_{a,b}M'_{b,c}}{M_{a,b}M_{b,c}}. \tag{I.115}$$

Taking now $b = c$ in the last equation, gives

$$\frac{M'_{a,a}}{M_{a,a}} = \frac{M'_{c,c}}{M_{c,c}}, \tag{I.116}$$

so that $a \mapsto \frac{M'_{a,a}}{M_{a,a}}$ is constant, say, equals to α. Replacing $\frac{M'_{a,a}}{M_{a,a}}$ by α in (I.115) gives (3).
To prove $(3) \Rightarrow (1)$, it may be useful to see (3) as an equation ruling the addition of any letter b between a and c. Let us add two letters u and v (or b and c) between a and d...

$$\alpha \frac{M'_{a,d}}{M_{a,d}} = \frac{M'_{a,v}M'_{v,d}}{M_{a,v}M_{v,d}} = \frac{M'_{a,c}M'_{c,d}}{M_{a,c}M_{c,d}} \Rightarrow \frac{M'_{a,u}M'_{u,v}M'_{v,d}}{M_{a,u}M_{u,v}M_{v,d}} = \alpha^2 \frac{M'_{a,d}}{M_{a,d}} = \frac{M'_{a,b}M'_{b,c}M'_{c,d}}{M_{a,b}M_{b,c}M_{c,d}}.$$

This gives (1).

It remains to prove the last statement. Using point (3), for a right $\alpha > 0$, for any word $x \in E_\kappa^n$

$$\frac{M'_{a,x_1}M'_{x_1,x_2}\cdots M'_{x_n,d}}{M_{a,x_1}M_{x_1,x_2}\cdots M_{x_n,d}} = \alpha \frac{M'_{a,x_2}M'_{x_2,x_3}\cdots M'_{x_n,d}}{M_{a,x_2}M_{x_2,x_3}\cdots M_{x_n,d}} = \cdots = \alpha^n \frac{M'_{a,d}}{M_{a,d}}.$$

Then multiplying by the LHS denominator and summing over all values of $x_1, \cdots, x_n$, we obtain

$$\frac{(M')^{n+1}_{a,d}}{(M)^{n+1}_{a,d}} = \alpha^n \frac{M'_{a,d}}{M_{a,d}}.$$

Since M and M' are positive recurrent, taking the limit when $n \to \infty$, we get

$$\frac{\rho'_d}{\rho_d} = \frac{M'_{a,d}}{M_{a,d}} \lim_{n\to\infty} \alpha^n, \tag{I.117}$$

□

where ρ and ρ' are the invariant measures for the Markov kernels M and M'. Hence $\alpha = 1$. Taking $b = c = a$ in (3) then gives $M'_{a,a}/M_{a,a} = 1$ for any a. Using this relation and taking $d = a$ and $\alpha = 1$ in (I.117), we obtain $\rho'_a = \rho_a$, and still from (I.117), this implies $M'_{a,d} = M_{a,d}$ for any (a, d).

I.C Proof of Theorem I.3.1.2

We start with a preliminary lemma.

Lemma I.C.0.1
(Analogue of Lemma I.5.1.1) If M is a positive Markov kernel with memory m, then for all $a, c \in E_\kappa^{[\![1,m]\!]}$, any function $f, g : E_\kappa^m \to \mathbb{R}$,

$$\sum_{b \in E_\kappa^L} f(a)g(c)\mathsf{Z}_w \prod_{i=1}^{L+m} M_{w[\![i,i+m]\!]} = 0$$

where in the sum, $w = w[\![1, L+2m]\!]$ is used instead of abc (meaning that $w[\![1,m]\!] = a$ and $w[\![L+m+1, L+2m]\!] = c$).

The proof is the same as that of Lemma I.5.1.1 taking into account Remark I.5.1.2.

The proof of Theorem I.3.1.2 is very similar to that of Theorem I.2.1.2; we discuss only the main differences.

We will prove the two cyclical implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i)$ and $(v) \Rightarrow (vi) \Rightarrow (vii) \Rightarrow (viii) \Rightarrow (ix) \Rightarrow (v)$.

• **Proof of** $(i) \Rightarrow (ii)$ The following comparison gives the result if we consider $n = 2k+1$ large enough

$$\mathsf{NLine}_n(x[\![1,k+1]\!]0^k) - \mathsf{NLine}_n(x[\![1,k]\!]0^{k+1}) = \rho_1 \mathsf{Replace}_{\mathsf{h}}(x[\![k+2-\mathsf{s},k+1]\!]0^{\mathsf{s}-1};0)$$

and the LHS is zero, because the solution is invariant.

• **Proof of** $(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$ The proof of Theorem I.2.1.2 may be adapted.

• **Proof of** $(v) \Rightarrow (i)$ The proof of the following lemmas can be adapted

Lemma I.C.0.2
(Analogue of Lemma I.5.1.3) Let M be a Markov kernel with memory m and positive entries. If $\mathsf{Master}_{\mathsf{h}}^{M,\mathsf{T}} \equiv 0$, then for all $n \geq 2L-1$, all $k \in [\![2, n-1]\!]$, all $x \in E_\kappa^n$,

$$\mathsf{NLine}_n^{\rho,M,\mathsf{T}}(x) = \mathsf{NLine}_{n-1}^{\rho,M,\mathsf{T}}(x^{\{k\}}).$$

Lemma I.C.0.3
(Analogue of Lemma I.5.1.4) If M is a positive Markov kernel with memory m then for all $n \geq m$, for all $x \in E_\kappa^{[\![1,n]\!]}$, then

$$\begin{aligned}\sum_{y \in E_\kappa} \mathsf{NLine}_{n+1}^{\rho,M,\mathsf{T}}(xy) M_{x[\![n-m+1,n]\!]y} &= \mathsf{NLine}_n^{\rho,M,\mathsf{T}}(x) \\ &= \sum_{y \in E_\kappa} \mathsf{NLine}_{n+1}^{\rho,M,\mathsf{T}}(yx[\![1,n]\!]) M_{x[\![n-m,n]\!]}.\end{aligned}$$

Moreover if $n \leq m-1$, then

$$\sum_y \mathsf{NLine}_{n+1}^{\rho,M,\mathsf{T}}(x[\![1,n]\!]y) = \mathsf{NLine}_n^{\rho,M,\mathsf{T}}(x[\![1,n]\!]) = \sum_y \mathsf{NLine}_n^{\rho,M,\mathsf{T}}(yx[\![1,n]\!]).$$

To prove this modification, just see that from Line to NLine we divided by $\prod_{j=1}^{n-m} M_{x[\![j,m+j]\!]}$ if $n \geq m+1$ and $\mathsf{NLine}_n^{\rho,M,\mathsf{T}} \equiv \mathsf{Line}_n^{\rho,M,\mathsf{T}}$ if $n \leq m$, hence summing over x_n (respectively x_1), using $\sum_{b \in E_\kappa} M_{a,b} = 1$ (respectively $\sum_{a \in E_\kappa^m} \rho_a M_{a,b} = \rho_b$) and Lemma I.C.0.1 gives the result.

Lemma I.C.0.4
(Analogue of Lemma I.5.1.5) Consider M a positive Markov kernel with memory m, and T a JRM with range L. If $\mathsf{Master}_{\mathsf{h}}^{M,\mathsf{T}} \equiv 0$ then for all n, $\mathsf{NLine}_n^{\rho,M,\mathsf{T}} \equiv 0$.

Proof. Since by Lemma I.C.0.3 one can deduce the nullity of $\mathsf{NLine}_n^{\rho,M,\mathsf{T}}$ from $\mathsf{NLine}_{n+1}^{\rho,M,\mathsf{T}}$, and since by Lemma I.C.0.2 we may deduce the nullity of $\mathsf{NLine}_n^{\rho,M,\mathsf{T}}$ from that of $\mathsf{NLine}_{n-1}^{\rho,M,\mathsf{T}}$ for $n \geq 2L-1$, it suffices to prove $\mathsf{NLine}_N^{\rho,M,\mathsf{T}} \equiv 0$ from the N of our choice, as far as it is larger than $\max\{2L-1, 2m\}$ (where $2m$ is chosen for commodity).

We will adapt the argument of Lemma I.5.1.5. The argument is a bit more involved here. Take $A \in E_\kappa^N$. Using iteratively Lemma I.C.0.3,

$$\sum_{b[\![1,m]\!] \in E_\kappa^m} \mathsf{NLine}_{N+m}^{\rho,M,\mathsf{T}}(Ab[\![1,m]\!]) \prod_{j=1}^m M_{A[\![N-(m-j),N]\!]b[\![1,j]\!]} = \mathsf{NLine}_N^{\rho,M,\mathsf{T}}(A). \quad \text{(I.118)}$$

By Lemma I.5.1.3, $\mathsf{NLine}^{\rho,M,\mathsf{T}}_{N+m}(Ab[\![1,m]\!])$ is unaffected by the suppression of inner letters (as long as it remains at least $2L-2$ letters), so that $\mathsf{NLine}^{\rho,M,\mathsf{T}}_{N+m}(Ab[\![1,m]\!]) = \mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A[\![N-m]\!]b[\![1,m]\!])$. Hence, (I.118) becomes

$$\sum_{b[\![1,m]\!]\in E_\kappa^m} \mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A[\![1,N-m]\!]b[\![1,m]\!]) \prod_{j=1}^{m} M_{A[\![N-(m-j),N]\!]b[\![1,j]\!]} = \mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A). \quad (I.119)$$

Consider the matrix $\Gamma = (\Gamma_{u,b})_{u,b\in E_\kappa^m}$ defined by

$$\Gamma_{u,b} = \prod_{j=1}^{m} M_{u[\![j,m]\!]b[\![1,j]\!]}$$

(I.119) is equivalent to

$$\sum_{B\in E_\kappa^m} \mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A[\![1,N-m]\!]B)\Gamma_{A[\![N-m+1,N]\!],B} = \mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A) \quad (I.120)$$

Now, $\Gamma_{u,b}$ is a Markov kernel: it is $\mathbb{P}(X[\![m+1,2m]\!] = b \,|\, X[\![1,m]\!] = u)$ for a Markov chain with memory m and kernel K. Therefore rewriting in (I.120), A under the form $A[\![1,N-m]\!]A'[\![1,m]\!]$ where $A'[\![1,m]\!]$ is the suffix of A, this equation is equivalent to

$$\mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A[\![1,N-m]\!]A'[\![1,m]\!]) = \sum_{B\in E_\kappa^m} \mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A[\![1,N-m]\!]B)\Gamma_{A'[\![1,m]\!],B}$$

from what appears that

$$v_{A[\![1,N-m]\!]} = \left[\mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A[\![1,N-m]\!]B), \quad B\in E_\kappa^m\right]$$

is a left eigenvector to the Γ^t. Taking into account the hypothesis on M,

$$v_{A[\![1,N-m]\!]} = \lambda_{A[\![1,N-m]\!]} \begin{bmatrix}1 & \cdots & 1\end{bmatrix}$$

which means that $\mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A[\![1,N-m]\!]B) = \lambda_{A[\![1,N-m]\!]}$ does not depend on B.

Since $N \geq 2L-1$, and since by Lemma I.5.1.3, $\mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A[\![1,N-m]\!]B)$ is unaffected by the suppression/addition of inner letters (as long as these operations are done on words with more than $2L-1$ letters for the suppression and $2L-2$ letters for the addition), for any $C\in E_\kappa^{N-2m}$

$$\begin{aligned}\mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A[\![1,N-m]\!]B) &= \mathsf{NLine}^{\rho,M,\mathsf{T}}_{2N-2m}(A[\![1,N-m]\!]CB)\\ &= \mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A[\![1,m]\!]CB)\end{aligned}$$

so that $\lambda_{A[\![1,N-m]\!]}$ depends only of the m first letters of A (we keep m letters for commodity, we could have kept only A_1). Hence, there exists a function f such that

$$\mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A[\![1,m]\!]B) = f(A[\![1,m]\!]), \quad (I.121)$$

for any word $B\in E_\kappa^{N-m}$. Now, we claim that for any $k\geq m$, $\mathsf{NLine}^{\rho,M,\mathsf{T}}_{k}(A[\![1,k]\!]) = f(A[\![1,m]\!])$. If $k\geq N$, this can be proved using the argument above. For $m\leq k\leq N$, by Lemma I.C.0.2, $\mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A[\![1,k]\!]) = \sum_{y[\![1,N-m]\!]} \mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A[\![1,k]\!]y[\![1,N-k]\!]) \prod_{j=1}^{N-k} M_{y[\![j-m,j]\!]}$ where $y_j = A_{k+j}$ for $j\leq 0$. Plugging that for any $y[\![1,N-k]\!]$, $\mathsf{NLine}^{\rho,M,\mathsf{T}}_{N}(A[\![1,k]\!]y[\![1,N-k]\!]) = f(A[\![1,m]\!])$, we get the result.

Now, consider the matrix $\mathbf{M} = (\mathbf{M}_{u,v})_{u,v\in E_\kappa^m}$ defined by

$$\mathbf{M}_{a[\![1,m]\!],b[\![1,m]\!]} = \mathbf{1}_{a[\![2,m]\!]=b[\![1,m-1]\!]} M_{a[\![1,m]\!],b_m}.$$

If $X = (X_k, k \in \mathbb{Z})$ is a Markov chain with kernel M and memory m, $\mathbf{M}$ is simply the Markov kernel of the Markov chain $(Y_k, k \in \mathbb{Z})$ defined by $Y_k = (X[\![k, k+m-1]\!])$. Hence,

$$\mathbb{P}(Y_m = b \mid Y_0 = a) = \prod_{j=0}^{m-1} M_{a[\![j,m-1]\!]b[\![0,j]\!]} = \mathbf{M}^m_{a,b}.$$

Let C be a word with m letters. Measuring the balance at C gives the relation, $\sum_{B\in E_\kappa^m} \mathsf{Line}_{2m}^{\rho,M,\mathsf{T}}(BC) = \sum_{B\in E_\kappa^m} \mathsf{Line}_{2m}^{\rho,M,\mathsf{T}}(CB)$ so that, by (I.51) and given that $\mathbf{M}$ is a Markov kernel

$$\begin{aligned}\sum_{B\in E_\kappa} f(B)\mathbf{M}^m_{BC} &= \sum_{B\in E_\kappa} \mathsf{NLine}_{2m}^{\rho,M,\mathsf{T}}(BC)\mathbf{M}^m_{BC}\\ &= \sum_{B\in E_\kappa^m} \mathsf{NLine}_{2m}^{\rho,M,\mathsf{T}}(CB)\mathbf{M}^m_{C,B} = f(C),\end{aligned}$$

for the f given in (I.121). Since $\mathbf{M}^m$ is positive, $f(C) = \alpha\rho_C$, where ρ_C is the invariant distribution of the Markov kernel $\mathbf{M}^m$.

It remains to check that $\alpha = 0$. For this, write

$$\sum_{B\in E_\kappa^m} \mathsf{Line}_m^{\rho,M,\mathsf{T}}(B) = \sum_{B\in E_\kappa^m} f(B) = \sum_{B\in E_\kappa^m} \alpha\rho_B = \alpha$$

and can be rewritten, taking into consideration Lemma I.C.0.3 and the discussion below it

$$\begin{aligned}\sum_{B\in E_\kappa^m} \mathsf{Line}_m^{\rho,M,\mathsf{T}}(B) &= \sum_{A\in E_\kappa^q} \rho_{A[\![1,m]\!]} \sum_{B\in E_\kappa^m} \sum_{C\in E_\kappa^q} \left(\prod_{\ell=1}^{2q-m} M_{w[\![\ell,\ell+m]\!]}\right)\\ &\quad\times \sum_j Z_{w[\![j+1,j+\mathsf{s}]\!]} 1_{[\![j+1,j+\mathsf{s}]\!]\cap[\![q+1,q+m]\!]\neq\varnothing}\end{aligned}$$

where $w[\![1, 2q+m]\!] = ABC$. The contribution of each j such that $[\![j+1, j+\mathsf{s}]\!]$ intersects $[\![q+1, q+m]\!]$, that is the indices of the letters of B, can be considered apart, and the summation of each index of w which does not enter in Z can be simplified. The contribution of the j-th term becomes

$$\sum_{w[\![j+1,j+\mathsf{s}]\!]} \rho_{w[\![j+1,j+m]\!]} \mathsf{Z}_{w[\![j+1,j+\mathsf{s}]\!]} \prod_{i=j+1}^{j+L+m} M_{w[\![i,i+m]\!]}$$

which is 0 by Lemma I.C.0.1. □

I.D Proof of Theorem I.3.1.5

First, the linearity principle (Remark I.2.1.10) can be applied to a larger L and M: if a (ρ, M)-Markov law is invariant by T, then $\mathsf{Cycle}_n^{M,\mathsf{T}} \equiv 0$ for any n.

For the converse, starting from $\mathsf{NCycle}_n^{M,\mathsf{T}} \equiv 0$ for $n \geq \kappa^m$ we want to prove that $\mathsf{NLine}_n^{\rho,M,\mathsf{T}} \equiv 0$ for any $n \geq 1$. If $n-m+1 \geq \kappa^m+1$, for any word w of size n, by the pigeon hole principle, there is a word w' of size m which appears twice as a factor of w. The sum on Z (indexed by ℓ letters) along each factors of size ℓ of w between the two occurrences of w' produce the same contribution as a cycle, and then can be simplified. After simplification, remains only the words w with at most κ^m+m letters. The number of remaining words after simplification is then finite. Therefore,

$$\left\| \frac{\partial}{\partial t}\mu_n^t \right\|_\infty \leq C := \sup_{N \leq \kappa^m+m} \sup_{x \in E_\kappa^N} \mathsf{Line}_N^{\rho,M,\mathsf{T}}(x) < +\infty.$$

The constant C does not depend on n, and therefore

$$d_{TV}(\mu_n^t, \mu_n^0) \leq Ct,$$

where d_{TV} denotes the total variation distance. At time $\varepsilon > 0$, for a fixed r, then we have $d_{TV}(\mu_r^\varepsilon, \mu_r^0) \leq C\varepsilon$. Let us prove that the mixture condition (irreducibility and aperiodicity) implies that it is in fact 0. Recall the following property of the distance in variation:

$$d_{TV}(\mu,\nu) = 2 \inf \mathbb{E}(\mathbf{1}_{X' \neq Y'})$$

where the infimum is taken over all couplings, that is, on all pairs (X', Y') where X' is μ distributed and Y' is ν distributed. Now, take two pairs $(X_1, X_2) \sim \mu_{1,2}$ and $(Y_1, Y_2) \sim \nu_{1,2}$ with independent marginals, where X_1 and X_2 are μ distributed, Y_1 and Y_2 are ν distributed. Suppose that (X_i, Y_i) for $i = 1, 2$ are optimal couplings for the marginals, that is $d := d_{TV}(\mu,\nu) = 2\mathbb{E}(\mathbf{1}_{X_1 \neq Y_1}) = 2\mathbb{E}(\mathbf{1}_{X_2 \neq Y_2})$. We have then

$$d_{TV}(\mu_{1,2}, \nu_{1,2}) = d^2 + 2d(1-d) = 2d - d^2 > (3/2)d,$$

where this last equality is valid when d is small ($d < 1/2$). Hence, if one knows that the distance $d_{TV}(\mu_{1,2}, \nu_{1,2}) < \varepsilon < 1/2$ and that the marginals are independent, then $d_{TV}(\mu,\nu) < (2/3)\varepsilon$.

The strategy is as follows: we will deduce from the inequality $d_{TV}(\mu_n^t, \mu_n^0) \leq Ct$ for any n, that $d_{TV}(\mu_n^t, \mu_n^0) \leq (3/4)Ct$ for any n, so that necessarily $d_{TV}(\mu_n^t, \mu_n^0) = 0$.

Take $I_r(k) = [\![1,r]\!] \cup [\![(k-1)r+1, kr]\!]$. Now write $d_{TV}(\mu_{I_r(k)}^t, \mu_{I_r(k)}^0) \leq d_{TV}(\mu_{[\![0,kr]\!]}^t, \mu_{[\![0,kr]\!]}^0) < \varepsilon$. Since $I_r(k)$ is the union of two intervals, $\mu_{I_r(k)}^t$ is (for a clear notation) the distribution of the pair $(X^t[\![1,r]\!], X^t[\![(k-1)r+1, kr]\!])$. According to the previous discussion, to conclude it suffices to prove that when $k \to +\infty$, the two pairs $A_t(k) := (X^t[\![1,r]\!], X^0[\![1,r]\!])$ and $B_t(k) := (X^t[\![(k-1)r+1, kr]\!], X^0[\![(k-1)r+1, kr]\!])$ converges to two independent variables with the same distribution.

By the hypothesis we made on the Markov kernel, for any $\varepsilon' > 0$, it is possible to find a k large enough such that the variation distance of the initial configuration $(X^0[\![1,r]\!], X^0[\![(k-1)r+1, kr]\!])$ with a pair of independent r.v. with the same marginals is smaller than $\varepsilon'' > 0$ for the ε'' of our choice.

The fact that the initial configuration converges to independent vectors with same distribution when $k \to +\infty$, is not sufficient. We need to show that their evolution till time t are asymptotically (in k) independent too.

The argument is routine: Since the number of colors is finite, L is finite, $\max_{w,w' \in E_\kappa^L} \mathsf{T}_{[w|w']} < +\infty$. For $t < \infty$ fixed we build a dependence graph G_t as follows: first, the vertex set of the graph is the set of intervals of size L. For each jump that has occurred in a interval I before time t we add an edge between this interval and the intervals which intersect it (to encode, the fact that the state at time t of these intervals may have been modified by the jump in I). Since $\max_{w,w' \in E_\kappa^L} \mathsf{T}_{[w|w']} < +\infty$, when $k \to +\infty$, the probability that the two intervals $[\![1, r]\!]$ and $[\![(k-1)r+1, kr]\!]$ intersect distinct connected components of G_t goes to 1. This suffices to deduce the asymptotic independence of $(A_t(k), B_t(k))$ when $k \to +\infty$.

II

Survival and coexistence for spatial population models with forest fire epidemics.

Contents

We investigate the effect on survival and coexistence of introducing forest fire epidemics to a certain two-species spatial competition model. The model is an extension of the one introduced by Durrett and Remenik [42], who studied a discrete time particle system running on a random 3-regular graph where occupied sites grow until they become sufficiently dense so that an epidemic wipes out large clusters. In our extension we let two species affected by independent epidemics compete for space, and we allow the epidemic to attack not only giant clusters, but also clusters of smaller order. Our main results show that, for the two-type model, there are explicit parameter regions where either one species dominates or there is coexistence; this contrasts with the behavior of the model without epidemics, where the fitter species always dominates. We also characterize the survival and extinction regimes for the particle system with a single species. In both cases we prove convergence to explicit dynamical systems; simulations suggest that their orbits present chaotic behavior.

II.1 Introduction

In the mathematical biology literature, resource competition between n species is widely modeled through Lotka-Volterra type ODEs of the form

$$\frac{dx_i(t)}{dt} = x_i(t)\Big(a_i - \sum_{j=1}^{n} b_{ij}x_j(t)\Big), \qquad i = 1,\dots,n$$

if time is taken to be continuous, and the analogous difference equations

$$x_i(m+1) - x_i(m) = x_i(m)\Big(a_i - \sum_{j=1}^{n} b_{ij}x_j(m)\Big), \qquad i = 1,\dots,n$$

if time is taken to be discrete, where $x_i \in [0,1]$ represents the density of the i-th species and the a_i's and b_{ij}'s are the parameters of the model. The term inside the parentheses determines the effect of inter-specific and intra-specific competition, and has the advantage of being simple enough for an easy interpretation of its coefficients while, at the same time, allowing the system to exhibit a rich asymptotic behavior, including fixed points, limit cycles and attractors. However, despite its ubiquitousness, the classical model seems inadequate to explain diverse and complex ecosystems, as conditions for stability become more restrictive for larger values of n; the same seems to be true regarding conditions for coexistence (see e.g. [65, 8]), implying that, unless the parameters have been finely tuned, most species will be driven to extinction as a result of competition.

Even though it has been argued that natural selection alone may be able to tune the relevant parameters to yield a coexistence regime [1], a considerable amount of effort has been directed towards extending models such as Lotka-Volterra in ways that favor coexistence. Extensions of this sort include, for example, the addition of predators [92, 66, 101], of random fluctuations in the environment [115, 86] and of diseases [67, 98]; these extensions succeed in promoting biodiversity, but result in much more complicated models. An alternative way of extending the model is based on questioning the linear form of the inter-specific and intra-specific competition terms; indeed, for large population densities the intra-specific competition of a species has an increasingly important non-linear component, known as the *crowding effect*, which is overlooked in the original equations. The crowding effect is capable of effectively outbalancing the inter-specific competition effect for a significantly larger set of parameters, permiting coexistence even when n is large [63, 102, 53].

One important source for the crowding effect is the fact that at high population densities the connectedness between individuals tends to be high, making it easier for an infectious disease to spread through the population and giving rise to epidemic outbreaks. To the best of our knowledge, the effect that this phenomenon may have on coexistence has not been explored in the setting of competing spatial population models. This provides the main motivation for this chapter.

The model which we will study is based on a particle system introduced by Durrett and Remenik [42], which we will refer to as the *moth model (MM)*. It is inspired by the gypsy moth, whose populations grow until they become sufficiently dense for the nuclear polyhedrosis virus (*Borralinivirus reprimens*, which strikes at larval stage and spreads between nearby hosts) to reduce them to a low level. The MM is a discrete time particle system which alternates between a growth stage akin to a discrete time contact process and a forest fire stage where an epidemic randomly destroys entire clusters of occupied sites. (Forest fire models, which were first introduced in [38], have received much interest as a prime example of a system showing self-organized criticality, see e.g. [97], but this is not the focus of this chapter). [42] was devoted mostly to the study of the evolution of the density of occupied sites in the limit as the size of the system goes to infinity. Its main result showed that the system converges to a discrete-time dynamical system which, for large enough rates of population growth, and as a result of the forest fire epidemic mechanism, is chaotic.

In this work we study an extension of the moth model to a case where there are multiple species competing for space, each one affected by a different disease. As expected, when birth rates are sufficiently large the evolution of the system still presents chaotic behavior. The main goal of this chapter is to show that, in the case of two species, the introduction of forest fire epidemics can promote coexistence. The intuition behind this phenomenon is simple. Suppose that we have two species competing for space, in a situation where we would expect the fitter species to drive the other one to extinction. If we introduce forest fire epidemics into the system then the fitter species, which in particular achieves higher densities, will be more susceptible to the destruction of very large occupied clusters. This will have the effect of periodically clearing space for the growth of the weaker species, which may then have a chance to survive.

An important feature of our study is the fact that it is done not only at the level of the limiting dynamical system, but also for the (finite) particle system, for which we show that, depending on the parameters of the model, the weaker species may die out quickly or it may coexist with the fitter species for a relatively long time. We perform an analogous analysis of survival for the one-species particle system, complementing the [42] result for the limiting dynamical system.

Since the MM provides the basic setting for all of our results, we will begin by introducing it and the main results of [42] in some detail, and defer an overview of our extension and results until Sections II.1.2 and II.1.3. The detailed discussion of our results will be postponed until Section II.2.

II.1.1 The moth model

Definition II.1.1.1
The MM is a discrete time Markov process $\left(\eta_k^N\right)_{k\geq 0}$ taking values in $\{0,1\}^{G_N}$, where G_N is a finite, possibly random graph of size N, in which each vertex x is either occupied by a particle ($\eta_k^N(x)=1$) or empty ($\eta_k^N(x)=0$). The dynamics of the process at each time step is divided into two consecutive stages, *growth* and *epidemic*:

Growth: Each particle gives birth to a mean $\beta>0$ number of individuals and then dies. Individuals born at site x are sent to a randomly chosen site in its *growth neighborhood* $\mathcal{N}_N(x)\subseteq G_N$.

Epidemic: Each site is attacked by an infection with probability α_N, independently across different sites. When an occupied site x is attacked, the infection wipes out the entire connected component of occupied sites containing x. The occupied sites which survive the epidemic are the ones making up the population at the start of the next time step.

The main goal of [42] was to show that, for suitable choices of graphs G_N, and under some growth conditions on $\mathcal{N}_N$ and α_N, the trajectories described by the densities

$$\rho_k^N=\frac{1}{N}\sum_{x\in G_N}\eta_k^N(x) \tag{II.1}$$

converge to the orbit of a deterministic dynamical system which, for certain parameter values, is chaotic. The dynamical system obtained in [42] is defined by a map $h : [0,1] \to [0,1]$ of the form $h = g_0 \circ f_\beta$, where

$$f_\beta(p) = 1 - e^{-\beta p}$$

is the expected population density after the growth stage starting with density p and $g_0(q)$ is the expected density of sites that survive the epidemic stage when it attacks a population with density q which is uniformly spread (i.e. distributed according to a product measure with this density). The particular form of f_β follows from approximating the spatially dependent model by its mean field version. The function g_0, on the other hand, depends heavily on the choice of the sequence of graphs G_N and the epidemic parameters α_N, which in [42] are assumed to be in the *weak epidemic regime* $\alpha_N \longrightarrow 0$, which implies that in the $N \to \infty$ limit the epidemic only attacks infinite connected components.

In the first part of [42], the authors take $\{G_N\}_{N\in\mathbb{N}}$ to be a sequence of random connected 3-regular graphs and work in the case of mean-field growth, where $\mathcal{N}_N(x) = G_N$ for all N. The mean-field assumption implies that after the growth stage the process looks like percolation on G_N, and since this graph looks locally like a 3-regular tree then one can hope to obtain explicit formulas: indeed, the probability that the root (and by consequence any vertex) is in an infinite component can be computed in terms of the survival probability of a binary branching process, and is given by

$$g_0(p) = \begin{cases} p & \text{if } p \leq \frac{1}{2}, \\ \frac{(1-p)^3}{p^2} & \text{if } \frac{1}{2} < p \leq 1. \end{cases} \tag{II.2}$$

Together with the above expression for f_β, this gives (see fig. II.1)

$$h(p) = g_0 \circ f_\beta(p) = \begin{cases} 1 - e^{-\beta p} & \text{if } 0 \leq p \leq a_0, \\ \frac{e^{-3\beta p}}{(1-e^{-\beta p})^2} & \text{if } a_0 < p \leq 1. \end{cases} \tag{II.3}$$

To keep the notation simple, in everything that follows we omit the dependence of h on the parameters of the model.

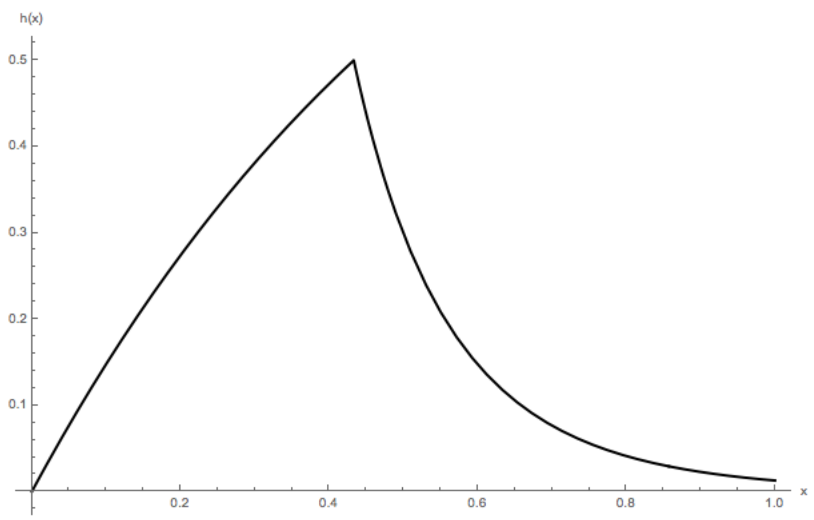


Figure II.1 – Plot h function, $\beta = 1.6$.

Throughout the chapter we will use the notation DS(h) to denote the dynamical system $\left(h^n(p)\right)_{n\geq 0}$ defined from the iterates h^n of a given map h.

The following theorem states the precise convergence result for the evolution of the density ρ_k^N of occupied sites as $N \to \infty$:

Theorem II.1.1.2: [42], Thm. 2
Suppose that $(G_N)_{N\in\mathbb{N}}$ is a sequence of random connected 3-regular graphs and that $\mathcal{N}_N(x) = G_N$ for all x and N. Assume that the infection probability of the epidemic satisfies $\alpha_N \longrightarrow 0$ and $\alpha_N \log_2(N) \longrightarrow \infty$ as $N \to \infty$, and also that $\rho_0^N \longrightarrow p \in [0,1]$ in distribution as $N \to \infty$. Then the process $(\rho_k^N)_{k\geq 0}$ converges in distribution as $N \to \infty$ (on compact time intervals) to the (deterministic) orbit of DS(h) started at p.

Key idea.

> The condition containing the $\log_2(N)$ appears in order to say that the whole process is close to the process ignoring epidemics coming from the outside of the local trees.

The behavior of DS(h) can be described as follows (see [42] for more details):

- If $\beta \leq 1$ then for every $p \in [0,1]$ the sequence $h^k(p)$ decreases to 0 as $k \to \infty$.
- If $\beta \in (1, 2\log 2]$ then the orbit of $h^k(p)$ eventually gets trapped inside the interval $[0, \frac{1}{2}]$, where $h \equiv f_\beta$, which means that there are no epidemic outbreaks. Inside this interval, $h^k(p)$ converges to the only positive fixed point of f_β.
- If $\beta > 2\log 2$ then the orbit of $h^k(p)$ is still trapped inside the interval $[h(\frac{1}{2}), \frac{1}{2}]$ but there is no longer convergence to a fixed point. Indeed, since $\beta > 2\log 2$, the fixed point of f_β is larger than $\frac{1}{2}$, so the successive growth stages drive the density above this value, at which time the epidemic kicks in and forces a relatively large jump back to $[h(\frac{1}{2}), \frac{1}{2}]$.

Thus the case $\beta \leq 1$ corresponds to the *extinction* regime (at least for the limiting dynamical system), while for all $\beta > 1$ we have $\liminf_{k\to\infty} h^k(p) > 0$ (for all $p \geq 0$), which corresponds to *survival*.

The next result establishes the chaotic behavior of the orbits of h when $\beta > 2\log 2$ (the third of the cases above, see fig. II.2):

Theorem II.1.1.3: [42], Theorem 1
The dynamical system DS(h) restricted to the interval $[h(\frac{1}{2}), \frac{1}{2}]$ is chaotic for every $\beta > 2\log 2$. Furthermore, if $\beta \in (2\log 2, 2.48]$, then the system has an invariant measure, $\mu = \mu \circ h^{-1}$, which is absolutely continuous with respect to the Lebesgue measure.

The notion of chaos in the first assertion of the theorem is the one given by Li and York [81] in their famous *period three implies chaos* theorem (see [42, Prop. 1.1] for more details). The authors also proved versions of Theorem II.1.1.2 and of the second assertion of Theorem II.1.1.3 (which is actually expected to hold for all $\beta > 2\log 2$) for the process running on the discrete torus with local growth, where newly born particles are sent to a local neighborhood with a diameter which grows suitably with N. However, in this case there is no explicit formula for g, nor numeric values for the critical parameters. It is precisely because of the availability of explicit formulas that, in everything that follows, we choose to work in the setting of random 3-regular graphs.

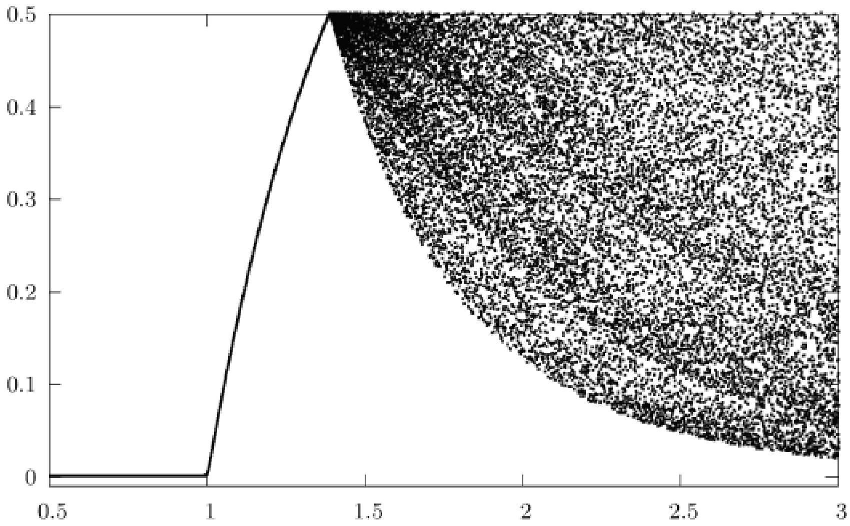


Figure II.2 – Orbits of the system $(h^k(p))_{k\geq 0}$ started at $p = 0.1$. The x-axis has the values of β used in the simulations, while the y-axis has $h^k(p)$ for $k = [\![501, 550]\!]$ (image by Daniel Remenik).

II.1.2 The Multi-type Moth Model

Our main interest in this chapter is the study of the *multi-type moth model (MMM)*, a natural extension of the moth model which considers multiple species competing for space subject to the same sort of epidemics. We describe it formally next.

Definition II.1.2.1
Fix a graph G_N as before and let $m \in \mathbb{N}$, which will be the number of species. The MMM is a discrete time Markov chain $\left(\eta_k^N\right)_{k\geq 0}$ taking values in $\{0, \ldots, m\}^{G_N}$; each site $x \in G_N$ can be occupied by an individual of type $i \in \{1, ..., m\}$ ($\eta_k^N(x) = i$) or vacant ($\eta_k^N(x) = 0$). The process depends on two sets of parameters, $\vec{\beta} = (\beta(1), \ldots, \beta(m)) \in \mathbb{R}_+^m$ and $\vec{\alpha}_N = (\alpha_N(1), \ldots, \alpha_N(m)) \in [0, 1]^m$, and as in the MM the dynamics of the process at each time step is divided into two consecutive stages:

Growth: An individual of type i at site $x \in G_N$ sends a Poisson$[\beta(i)]$ number of descendants to sites chosen uniformly at random in $\mathcal{N}_N(x) \subseteq G_N$. If a site receives individuals of more than one type, then the individual that survives is chosen uniformly among the individuals it receives (this fixes the type).

Epidemic: Each site x occupied by an individual of type i after the growth stage is attacked by an epidemic with probability $\alpha_N(i)$, independently across sites. The individual at x then dies along with its entire connected component of sites occupied by individuals of type i. This happens independently for $i = 1, \ldots, m$.

Note that we have assumed that the offspring of each individual is Poisson distributed. Although it would be possible to work with more general offspring distributions, as in the MM, we opt to make this assumption in order to simplify the presentation and proofs.

If one suppresses the epidemic stage then our process turns into a multi-type contact process, for which it is relatively easy to prove that the fitter species (i.e. the one with the largest growth parameter $\beta(i)$) will outcompete and drive to extinction all the other ones. In our main result, Theorem II.2.3.5, we show that the introduction of forest fire dynamics changes this picture, allowing two species to coexist even when they have different fitnesses. We remark, however, that in our model we are assuming that epidemics affect each species independently; this is natural when considering epidemics lacking cross-species transmission due to genetic distance, but is not a very realistic assumption if one thinks about the competition of different species of trees and takes the forest fire metaphor literally. It seems, nevertheless, that this assumption is important for coexistence to arise in our setting, as we will discuss further in Section II.2.2, where we present an example with non-specific epidemics in which the stronger species drives all the rest to extinction. It should be noted that this qualitative difference between epidemics with and without cross-species transmission is somewhat similar to the one found in the literature for predators, where the addition of a "specialist" predator to Lotka-Volterra systems can be more effective in promoting coexistence than the addition of a "generalist" predator (see [101]).

Remark II.1.2.2
A related model was studied by Chan and Durrett [24], who proved coexistence for the two-type, continuous time contact processes in $\mathbb{Z}^2$ with the addition of a different type of forest fires, which act by killing all individuals (regardless of their type, and regardless of whether they are connected) within blocks of a certain size. They showed that if the weaker competitor has a larger dispersal range then it is possible for the two species to coexist in the model with forest fires; this contrasts with Neuhauser's result [94] for the model without forest fires for which such coexistence is impossible. Our context is different, since we work on a random graph with forest fires which travel only along neighbors of the same type and which have an unbounded range, and since all species use the same dispersal neighborhoods The techniques we use are also different, and the results we obtain are of a slightly different nature. But the motivation is similar, and our results complement nicely with theirs.

As we already mentioned, all of our results will be proved in the case where G_N is a random connected 3-regular graph. The first step in our analysis of the MMM is an adaptation of Theorem II.1.1.2 to the multi-type case, Theorem II.2.1.2. More precisely, let $\{\rho_k^N\}_{k\geq 0}$ denote the sequence of density vectors obtained from $\{\eta_k^N\}_{k\geq 0}$ as

$$\rho_k^N = (\rho_k^{N,(1)}, \ldots, \rho_k^{N,(m)}) \qquad \text{with} \quad \rho_k^{N,(i)} = \frac{1}{N} \sum_{x \in G_N} \mathbb{1}_{\{\eta_k^N(x)=i\}}. \tag{II.4}$$

Then under suitable conditions on $\vec{\beta}$, $\vec{\alpha_N}$ and the growth neighborhood $\mathcal{N}_N(x)$, and assuming further that $\vec{\alpha_N}$ converges to some limiting $\vec{\alpha} \in [0,1]^m$, we show that ρ_k^N converges in distribution to a certain dynamical system DS(h) which is an m-dimensional analogue of the dynamical system we obtained for the MM.

We remark that in this work we are going beyond the weak epidemic regime of [42] by allowing the infection rates $\alpha_N(i)$ to converge to arbitrary values $\alpha(i) \in [0,1]$. This is natural from the biological point of view, as it incorporates into the model the effect of diseases with a fixed incidence rate. This generalization, which has the effect of modifying the $m \geq 1$ analog of g_0 (see (II.2)), has a major impact on the dynamical system, allowing the epidemics to kill not just infinite connected components but finite ones as well. In particular, for $\alpha(i) > 0$ the density of the type-i population no longer needs to be above the percolation parameter of the network for the epidemic to kick in, so we observe its effects at all times.

II.1.3 Overview of the main results

Phase diagram of the dynamical system

Since our main interest is to understand whether the introduction of forest fire epidemics can promote coexistence, for simplicity we restrict our study of the phase diagram of DS(h) to the case of two species ($m = 2$). Our results (Theorems II.2.3.1 and II.2.3.3) show that, as expected, there exist parameter regions where domination occurs (that is, where the fittest species drives the other one to extinction) as well as other parameter regions where both species coexist (that is, where both coordinates of h^k remain bounded below as $k \to \infty$). The regions obtained in our theorems are defined through two explicit inequalities, (II.13) and (II.14), which are naturally expressed in terms of the parameters

$$\phi_i := (1 - \alpha(i))\beta(i),$$

which we will refer to as the *fitness* of each species (and corresponds to the effective birth rate of individuals after considering the probability that a newly born particle does not survive the epidemic stage due to an infection arising in its location). In particular, we find the following (see also Figure II.8); here we assume that type 2 corresponds to the fitter species:

- Extinction is certain for any species with fitness value $\phi_i \le 1$. This is analogous to the extinction for the case $\beta \le 1$ described in the analysis of DS(h) for the MM.
- For every given fitness value $\phi_2 > 1$ of the stronger species we can choose ϕ_1 sufficiently close to, but larger than, 1, so that type 2 dominates.
- For any $\varepsilon > 0$ small we can choose ϕ_1 and ϕ_2 large but with relative fitness $\frac{\phi_1}{\phi_2} = \varepsilon$ such that both species coexist.

Note that, in view of the second and third points above, given any small $\varepsilon > 0$ we can choose two different sets of parameters with the same relative fitness ε so that in one case type 1 is driven to extinction while in the other case there is coexistence. Hence relative fitness does not provide enough information about the behavior of the system, which indicates that the effect of the forest fire epidemics is what is driving the qualitative difference in behavior.

As we have mentioned, even in the case of $m = 1$ our model provides an extension of the model studied in [42], as it drops the weak epidemics assumption by allowing for $\alpha_N \to \alpha > 0$. This extension is far from trivial at the level of the limiting dynamical system DS(h): as we will notice in Section II.2.2, from numerical simulations it is clear that for each fixed $\alpha \in (0, 1)$ the bifurcation diagram of DS(h) develops *bifurcation cascades* (also known as *period-doubling bifurcations*) in β, such as those seen for example for the quadratic maps $x \longmapsto rx(1 - x)$, see Figure II.4. See Section II.2.2 for more details.

Coexistence and survival for the particle systems

Our main results concern the behavior of the particle system for finite N and for one and two species. The main idea is to show that the behavior of the limiting dynamical systems DS(h) provides a good guide for the behavior of the original process. Note, however, that the MMM is a finite state Markov chain for which the extinction time of all types is almost surely finite, so we need to change our notions of coexistence and survival when working at the level of the particle systems. To this end we follow the usual approach (see e.g. [28, 41]) where one characterizes the different phases of the system in terms of the behavior of the (random) extinction times as a function of the network size N. Our main result in the case with $m = 2$ is Theorem II.2.3.5, which shows that there are parameter choices so that the weaker species may die out quickly while the fitter one survives for a relatively long time, and other parameter choices for which both species survive for a relatively long time.

The main challenge in proving results for our particle systems comes from the slow convergence of the empirical densities to the limiting dynamical system. This is intrinsic in the very nature of our model: it is hard to obtain a fine control on the distance between the finite system and its limit when the limiting system itself presents chaotic behavior, which makes it essentially impossible to predict its evolution. As a consequence, in our proof of coexistence we are not able to show that the extinction times of both species grow exponentially in N, as should be expected. For the case $m = 1$ (Theorems II.2.2.2 and II.2.2.3), on the other hand, we prove survival (when $\phi > 1$) and extinction (when $\phi \leq 1$) arguing directly on the particle system η_k^N (and not relying on the convergence to the dynamical system), and as a result we are able to prove that the expected extinction time does indeed grow exponentially at least for $\phi > \phi^*$ for some $\phi^* > 1$.

Outline

The rest of the chapter is organized as follows. In Section II.2.1 we state our convergence results (discussed in Section II.1.3). In Section II.2.2 we state the results related to the MM (discussed in Section II.1.3), while in Section II.2.3 we state the results related to the MMM (the multi-type case discussed in Section II.1.3). Last two sections also contain brief discussions about the main aspects involved in the proofs of our results. The proof themselves are deferred to Sections II.4, II.5 and II.5.3, devoted to the MM, the MMM and some technical results respectively.

II.2 Results

II.2.1 Convergence

As discussed in Section II.1.2, the starting point of our work is a convergence theorem for the MMM, analogous to the convergence proved in [42, Thm. 2] for the MM with weak epidemics. Analogously to (II.3), the limiting dynamical system will be given as DS(h) with h of the form $g_{\vec{\alpha}} \circ f_{\vec{\beta}}$, where $f_{\vec{\beta}}$ and $g_{\vec{\alpha}}$ describe the limiting densities after the growth and epidemic stage, respectively. In order to derive a good candidate for $f_{\vec{\beta}}$ we will focus for simplicity on the mean-field model ($\mathcal{N}_N(x) = G_N$), even though our result will be slightly more general, allowing for $\mathcal{N}_N(x) = B(x, r_N)$ for r_N converging to infinity sufficiently fast. Recalling the Poisson assumption on the offspring distribution, the expected proportion of occupied sites after the growth stage with initial densities given by $p \in S(m)$ is $1 - e^{-\sum_{i=1}^m \beta(i)p_i}$, and since in the process we let each site choose its type uniformly at random from the particles it receives, the expected density of sites occupied by type i after the growth stage is given by

$$f_{\vec{\beta}}^{(i)}(p) = \left(1 - e^{-\sum_{i=1}^m \beta(i)p_i}\right) \frac{\beta(i)p_i}{\sum_{i=1}^m \beta(i)p_i}. \tag{II.5}$$

The function $g_{\vec{\alpha}}$, on the other hand depends heavily on the particular choice G_N which, we recall, we always take to be a random 3-regular connected graph. In this case, and as explained in Section II.1.1, the graph looks locally like a 3-regular tree, so in order to guess a candidate for $g_{\vec{\alpha}}$ we can pretend that the epidemic stage acts on the infinite 3-tree $\mathcal{T}$. Let us also assume for a moment that $m = 1$. Then we need to analyze the effect of the epidemic when attacking a configuration of particles distributed as site percolation on $\mathcal{T}$ with a given density q (whose distribution, i.e. a product measure on $\{0,1\}^{\mathcal{T}}$ where each vertex is occupied with probability q, we denote as $\mathbf{P}_q$). Note that if $\mathcal{C}_r$ denotes the connected component of occupied sites containing r then the probability that r survives is given by $(1-\alpha_N)^{|\mathcal{C}_r|}\mathbb{1}_{\{|\mathcal{C}_r|>0\}}$. As a consequence, we should expect that the limiting probability that a given site is occupied after the epidemic stage (when it attacks a system with a fraction q of occupied sites) be given by

$$g_\alpha(q) = \mathbf{P}_q(r \text{ is occupied}, r \text{ survives the epidemic}) = \mathbf{E}_q((1-\alpha)^{|\mathcal{C}_r|}\mathbb{1}_{\{|\mathcal{C}_r|>0\}})$$

(here r is any vertex of $\mathcal{T}$).

The right hand side can be computed explicitly:

Proposition II.2.1.1
For any $q \in [0,1]$,

$$g_\alpha(q) = \begin{cases} 0 & \text{if } \alpha = 1, \\ \frac{\left(1-\sqrt{1-4(1-\alpha)q(1-q)}\right)^3}{8(1-\alpha)^2q^2} & \text{if } \alpha \in (0,1), \\ q & \text{if } \alpha = 0. \end{cases}$$

Key idea.

> Computation of the generating function of the number of subtrees containing the root in a 3-regular infinite tree.
> **Tools:** formal generating function

The explicit formula in the case $\alpha \in (0,1)$ (whose simple proof is included in Section II.3) is related to the generating function of the Catalan numbers. Now in the general case, when $m \geq 1$, since the epidemics attack each species independently, we deduce that the density of sites occupied by type i after the epidemic stage acts on a population with initial densities $\vec{q} \in S(m) := \{\vec{x} \in [0,1]^m : \sum_{i=1}^m x_i \leq 1;\ x_i \geq 0\}$ should be given by

$$g^{(i)}_{\vec{\alpha}}(\vec{q}) = g_{\alpha(i)}(q_i). \tag{II.6}$$

We are ready to state our main convergence result. Given $p \in S(m)$ define $h(p) = \big(h_1(p), \dots, h_m(p)\big)$ through

$$h_i(p) = g_{\alpha(i)} \circ f_{\vec{\beta}}(p).$$

Note that in the case $m = 1$, h_1 coincides with the function h defined above for the MM, which justifies our use of the same notation in both cases.

Theorem II.2.1.2
Consider the MMM with m types and with $\mathcal{N}_N(x) = B(x, r_N)$. Suppose that the sequences $\vec{\alpha_N}$ and r_N satisfy

$$\alpha_N(i) \xrightarrow[N\to\infty]{} \alpha(i) \in [0,1], \quad \alpha_N(i) r_N \xrightarrow[N\to\infty]{} \infty, \quad \text{and} \quad r_N \leq \tfrac{1}{25}\log_2(N)^2\alpha_N(i) \quad \forall N \in \mathbb{N}. \tag{II.7}$$

Suppose also that η_0^N is a product measure where each site is independently chosen to have type i with probability p_i. Then as $N \to \infty$, the density process $\big(\rho_k^N\big)_{k\geq 0}$ associated to the

MMM converges in distribution (on compact time intervals) to the deterministic orbit, starting at $p = (p_1, \ldots, p_m)$, of the dynamical system DS(h).

Key idea.

We propose a candidate limiting dynamical system and then we prove that, in fact, it is the limit. The technical assumption

$$\sqrt{\frac{r_N}{\alpha_N(i)}} \leq \frac{\log_2(N)}{5} \quad \forall N \in \mathbb{N}$$

helps to create a close modified version of the process at each point ignoring the epidemics from the outside of its ball of radius $\sqrt{r_N/\alpha_N(i)}$. Since 3-random regular graphs look like a 3-regular tree in a neighborhood of radius $\frac{\log_2(N)}{5}$, we can guess the limit candidate from ordinary generating functions of 3-regular trees.

Tools: Ordinary generating function, percolation on vertices.

The condition $\vec{\alpha}_N \to \vec{\alpha} \in (0,1)^m$ implies that in the limit the epidemic attacks not only giant clusters, but of small order too.

It is important to remark that even though, the diameter of a uniform random 3-regular graph is $O(\log(N))$ [16], the condition on r_N is not trivial (i.e. it is not of the order of the diameter):

- If $\alpha(i) \in (0,1]$ for all $i \in \{1, 2, \ldots, m\}$, the conditions on r_N simplifies and we can consider any $(r_N)_{N\in\mathbb{N}}$ going slowly enough to infinity. Ex: $r_N = \log(\log(N))$.
- If $\alpha(i) = 0$, then the MMM with parameters $\alpha_N(i) = \log(\log(\log(N)))^{-1}$ and $r_N = \log(\log(N))$ satisfies the hypothesis of the theorem.

Note that the last two assumptions in (II.7) are trivially satisfied in the mean-field case $\mathcal{N}_N(x) = G_N$ (i.e. $r_N = \infty$) if $\alpha(i) > 0$. The proof of Theorem II.2.1.2 is based on a relatively simple adaptation of the arguments of [42], needed to control the effect of the epidemic on finite components in order to go beyond the weak epidemics regime. It is worth noting (and will be clear from the proof) that in the mean-field case one could drop the product measure assumption on the initial condition (simply because the growth step returns a product measure anyway).

As discussed in the introduction, the behavior of the limiting dynamical system DS(h) provides us with informed guesses regarding the behavior of our particle systems. However, the above convergence result is not sufficient in order to prove that the behavior of the dynamical systems is in fact mirrored at the level of the finite MMM particle system; this requires quantitative estimates on the speed of convergence with an explicit control on the dependence on N. The approximation result that follows provides the necessary estimates in the case of mean-field growth. We believe that the result holds in the local growth setting of Theorem II.2.1.2; however, the algebraic expressions involved become even more complicated, so for simplicity we choose, here and in basically all the other upcoming results, to restrict the discussion to the simpler mean-field setting.

Let

$$\theta_\alpha(N) = \begin{cases} e^{-\sqrt{\log(N)}} & \text{if } \alpha = 0 \\ N^{-\alpha/5} & \text{if } \alpha > 0. \end{cases}$$

Theorem II.2.1.3

Consider the mean-field MMM (i.e. $\mathcal{N}_N(x) = G_N$ for all $x \in G_N$) with m types. Suppose that the sequence $\vec{\alpha}_N$ converges to some $\vec{\alpha} \in [0,1]^m$ and satisfies

$$-\alpha_N(i) \log N / \log \alpha_N(i) \longrightarrow \infty \tag{II.8}$$

for each i. Then for all $\delta > 0$ and $k \in \mathbb{N}$ there is a constant $C > 0$ depending only on δ and k such that for all $N \in \mathbb{N}$ and any initial condition η_0^N we have

$$\mathbb{P}\left(\left\|\rho_k^N - h^k(\rho_0^N)\right\| > \delta\right) \leq C\theta_{\underline{\alpha}}(N), \tag{II.9}$$

where $\underline{\alpha} = \min\{\alpha(1), \dots, \alpha(M)\}$.

Key idea.

> The main ingredient in the proof is Lemma II.3.0.2, which uses a comparison with a branching process to estimate the difference between h and the expected density after one step.

II.2.2 Results for the one-type model

Phase diagram and bifurcation cascades

We begin our study of the MM by briefly exploring the behavior of the limiting dynamical system, see fig. II.3. Recall our definition of the fitness parameter

$$\phi = \phi(\alpha, \beta) = \beta(1 - \alpha).$$

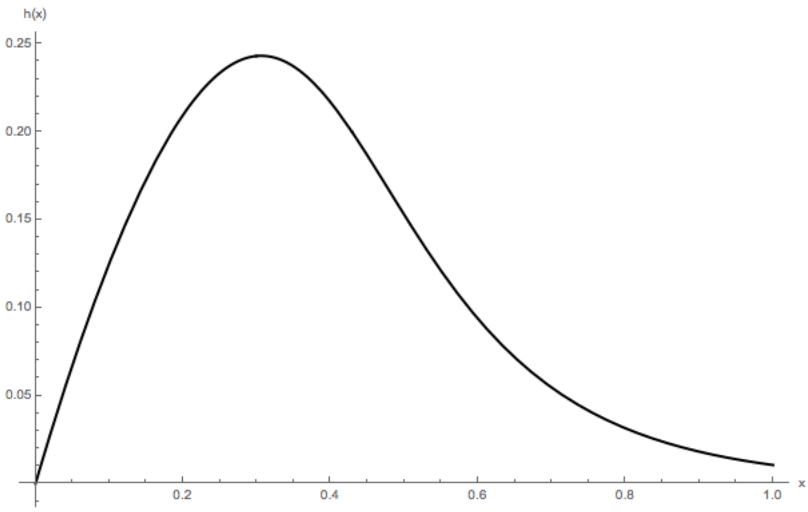


Figure II.3 – Plot h function, $\beta = 1.6$, $\alpha = 0.1$ ($\phi = 1.44$).

The following simple result establishes the desired phase transition between extinction and survival in the orbits of DS(h).

Proposition II.2.2.1
Let $\alpha \in [0,1]$ and $\beta > 0$.

(i) (Extinction) If $\phi(\alpha, \beta) \leq 1$, then

$$h^k(p) \xrightarrow{k\to\infty} 0 \qquad \forall p \in [0,1].$$

(ii) (Survival) if $\phi(\alpha, \beta) > 1$, then

$$\liminf_{k\to\infty} h^k(p) > 0 \qquad \forall p \in (0,1).$$

Key idea.

We study the behavior of repulsion and attraction of 0 as fixed point of the function h, which is related to the behavior of the derivative of h (similar, for example, to the case of Galton-Watson trees).

Let us briefly comment on an interesting behavior which becomes apparent from numerical simulations of the orbits of DS(h): the *bifurcation cascades* which we mentioned in Section II.2.2. These are sequences of period doubling bifurcations that occur as the parameter β is increased (for fixed $\alpha > 0$), and which accumulate at a certain finite value of β. Figure II.4 (left) shows bifurcation diagrams for DS(h) which clearly suggest the occurrence of this phenomenon in our system. This behavior contrasts with case $\alpha = 0$ where, as pointed out in [42] (see the discussion preceding Prop. 1.1 there), the system proceeds directly from a stable fixed point to a chaotic phase, without passing through period-doubling bifurcation; the parameter α has thus the effect of modulating the appearance of these bifurcation cascades.

The prototypical example of a dynamical system presenting this behavior is the one defined by the quadratic map $x \longmapsto rx(1-x)$, which has a first period doubling bifurcation occurring at $r = 3$ and then subsequent ones which continue up to $r \approx 3.56$, where a chaotic regime arises; this pattern is then repeated for larger values of r. This intricate behavior has been intensely studied since at least the 1970's, and presents an intriguing form of universality, which roughly states that the ratio of the gaps between subsequent period doubling bifurcations converges to a universal constant for a wide class of dynamical systems showing this type of cascades (see e.g. [47, 111], where several universality conjectures were settled). This area of dynamical systems continues to be developed to this day (see e.g. [99, 69]); we refer the reader to [112] for a nice account. Our simulations suggest that cascades appear for all $\alpha \in (0,1)$ when β is increased above 1, but proving this appears to be difficult due to the algebraic structure of h (in particular, the bifurcation points do not have a simple analytic expression). Figure II.4 (right) shows a simulation of the evolution of the MM for finite N and different values of β; note how some of the period doubling bifurcation behavior of the limiting system is still apparent in these simulations.

Figure II.5 presents a schematic summary, partly based on simulations, of the behavior of the orbits of DS(h) as a function α and β.

Extinction and survival for the particle system

We turn now to the dichotomy between extinction and survival at the level of the MM particle system for finite N. As discussed in the introduction, we will exhibit contrasting behaviors for the *absorption time*

$$\tau_N := \inf\left\{k \geq 1 : \eta_k^N(x) = 0 \forall x\right\} = \inf\left\{k \geq 1 : \rho_k^N = 0\right\}.$$

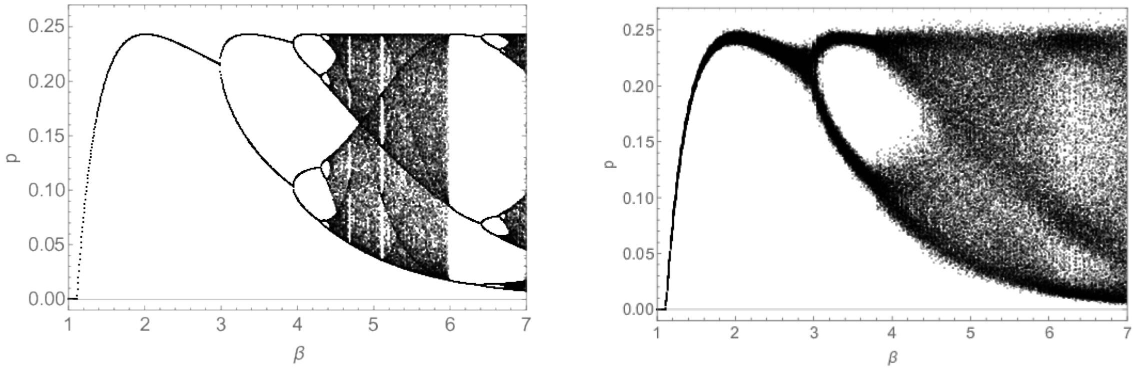


Figure II.4 – Left: Bifurcation diagram in β for DS(h) with $\alpha = 0.1$, showing the orbits of the system between iterations 900 and 1000 in the vertical direction for different values of β.
Right: Simulation of the evolution of the mean-field MM for $\alpha = 0.1$ and different values of β, from iteration 900 to 1000. Here $N \in \{20000, 40000, 100000\}$ (depending on β).

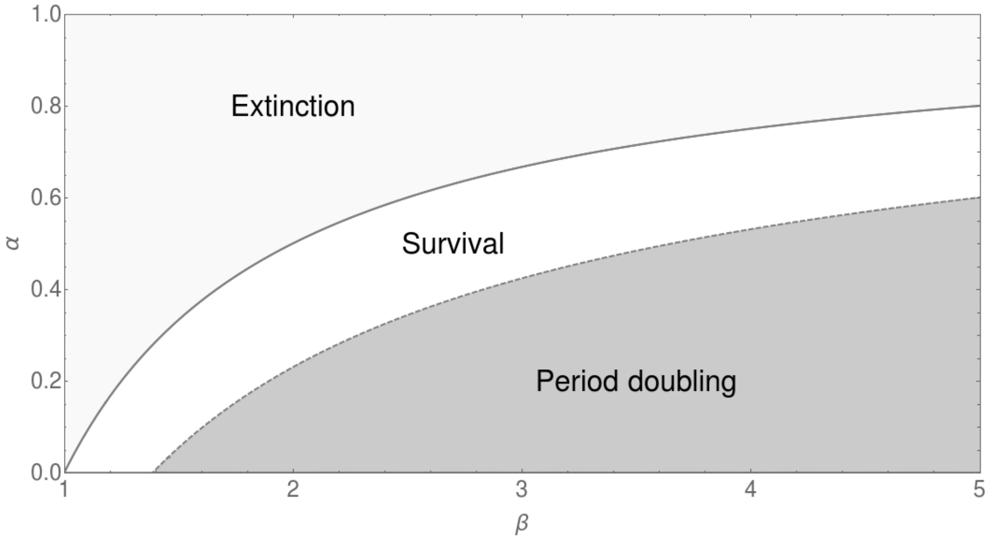


Figure II.5 – Approximate phase diagram of DS(h). The transition between extinction and survival is justified by Proposition II.2.2.1, while the one governing the appearance of bifurcation cascades (dashed line) is based on simulations.

The following result is satisfied for any fixed N (the size of the graph G_N) and any fixed choice of $\alpha(N)$, but for the sake of concreteness one may think of the case $\alpha(N) \to \alpha$ (or even with $\alpha(N) = \alpha$).

Theorem II.2.2.2
For the mean-field MM and any $N \in \mathbb{N}$ we have:

(i) (Extinction) If $\phi(\alpha_N, \beta) \leq 1$, then for all $n \in \mathbb{N}$ and any initial density ρ_0^N

$$\mathbb{P}(\tau_N \geq n) \leq \begin{cases} 1 - (1 - \phi(\alpha_N, \beta)^n)^N & \text{if } \phi(\alpha_N, \beta) < 1, \\ 1 - \left(1 - \frac{2}{n(1-\alpha_N)(\sigma^2 + \alpha_N \beta^2)}\right)^N & \text{if } \phi(\alpha_N, \beta) = 1, \end{cases}$$

where σ^2 is the offspring variance distribution of each particle in the growth stage. In particular, it follows that when $\phi(\alpha_N, \beta) < 1$ there is $C > 0$ independent of N such that

$$\mathbb{E}(\tau_N) \le C \log(N). \tag{II.10}$$

(ii) (Survival) If $\phi(\alpha_N, \beta) > 1$ and $\rho_0^N \ge \bar{\rho}_0$ for some $\bar{\rho}_0 > 0$, then there exists $c > 0$ (depending only on $\bar{\rho}_0$ and α_N) such that

$$\mathbb{P}(\tau_N \ge n) \ge \left(1 - \frac{c}{N}\right)^{3n}.$$

In particular, if we assume that $\alpha_N \log_2(N) \to \infty$ then

$$\mathbb{E}(\tau_N) \ge \frac{N}{4c}. \tag{II.11}$$

Key idea.

> The proof of extinction is based on a comparison with a branching process where one considers that epidemics do not spread. For survival we keep track of isolated occupied sites, which are not affected by epidemic events coming from other sites.
> **Tools:** Galton-Watson process, stochastic domination, coupling, Tchebychev inequality.

We believe that in the extinction regime the process actually has exponential expected absorption times. In the next result we show that this is indeed the case, at least for large enough ϕ, under an additional (but reasonable) condition on our random graphs.

Recall that a k-independent set of a graph G is a subset I of its vertices such that, for any $x, y \in I$, $d_G(x, y) \ge k$. Given $0 \le b < 1$ we define the events

$$\mathcal{R}_N(b) := \{G_N \text{ has a 3-independent set } I \text{ with } |I| \ge bN\}.$$

From [9, Thm. 1.1] we have there exists $b_1 \approx 0.09$ such that

$$\mathbb{P}(G_N \in \mathcal{R}_N(b_1)) \xrightarrow[N\to\infty]{} 1. \tag{II.12}$$

In words, our random 3-regular graphs contain a 3-independent set made out of fraction of at least b_1 of its vertices with probability close to 1 as N becomes large. This justifies conditioning on $\mathcal{R}_N(b_1)$ in the coming theorem.

Theorem II.2.2.3
Fix b_1 as in (II.12) and assume that $\phi(\alpha_N, \beta) > 1/b_1$ and that $\rho_0^N \ge \bar{\rho}_0$ for some $\bar{\rho}_0 > 0$. Then (in the case of mean-field growth) there is a $c > 0$, depending only on α_N and $\bar{\rho}_0$ such that

$$\mathbb{P}(\tau_N \ge n | G_N \in \mathcal{R}_N(b_1)) \ge (1 - \exp(-cN))^{3n}.$$

In particular, if $\alpha_N \longrightarrow \alpha \in [0, 1)$ and $\alpha_N \log_2(N) \longrightarrow \infty$ as $N \to \infty$, then

$$\mathbb{E}(\tau_N | G_N \in \mathcal{R}_N(b_1)) \ge \begin{cases} 3\exp(cN) & \text{if } \alpha \in (0,1), \\ 3\exp\left(\frac{cN}{\log(N)^2}\right) & \text{if } \alpha = 0. \end{cases}$$

Key idea.

The difficulty in Theorem II.2.2.2 is the dependence between vertices. This force us to use Chebychev inequality, which gives a polynomial bound. Here we give a setting with independence between a subset of sites, we then use a Chernoff type bound to obtain the exponential expected absorption time.
Tools: Chernoff bounds, independent sets of graphs.

II.2.3 Results for the multi-type model

In everything that follows we only consider the two-type case, $m = 2$.

Phase diagram

As for the MM, we begin by studying the behavior of the orbits of DS(h). The analysis is much more involved than the one for the one-type model, but it provides us with a glimpse on the role that the forest fire dynamics can have in aiding coexistence. In fact, our results in this part, together with the above approximation result (Theorem II.2.1.3), will constitute the basic ingredients for our later analysis of the particle system. As we mentioned, we will only look at the two-type case $m = 2$.

We are interested in identifying two different regimes for DS(h): we say that there is *domination* if one species goes extinct while the other one survives, i.e. if $(\liminf_{k\to\infty} h_1^k(\vec{p}), \liminf_{k\to\infty} h_2^k(\vec{p}))$ has one and only one vanishing coordinate, while we say that there is *coexistence* if both types survive, i.e. if the same liminf is strictly positive in both coordinates. Notice that once one species dies out, the behavior of the other one, say the one with type i, evolves according to the dynamical system given by $h_i = g_{\alpha(i)} \circ f_{\beta(i)}$ as in the one-type case.

Since we are interested in coexistence, we will restrict the discussion to the case when

$$\phi_i := \phi(\alpha(i), \beta(i)) > 1$$

for both $i = 1$ and $i = 2$; by Proposition II.2.2.1 we know that if this fails then at least one of the species would die out even when facing no competition, whence it easily follows that coexistence would be impossible. For concreteness we will always assume type 2 is fitter than type 1, i.e. $\phi_1 < \phi_2$.

In order to ease notation, from now on we denote, for a given initial condition $p \in [0, 1]^2$ and any $i \in \{1, 2\}$

$$p_i^k = h_i^k(p).$$

Theorem II.2.3.1: Coexistence
There is a continuous, increasing function $z\colon [0, 1] \longrightarrow \mathbb{R}^+$ (defined in (II.52)) satisfying $z(0) = 2\log(2)$ and $z(1) < 4\log(2)$ such that the following holds. Suppose that $\phi_2 > z(\alpha(2))$ and

$$\phi_1 \sqrt{\frac{2(1 - e^{-\kappa_2})(1 - e^{-\frac{\phi_2}{2}})}{\kappa_2 \phi_2}} \quad > \quad 1, \tag{II.13}$$

where κ_2 is the solution of $\kappa_2 = \beta(2)\, g_{\alpha(2)}(1 - e^{-\kappa_2})$. Then for any initial condition $p \in (0, 1)^2$ we have

$$\liminf_{k\to\infty} p_1^k > 0, \qquad \text{and} \qquad \liminf_{k\to\infty} p_2^k > 0.$$

Key idea.

The main idea behind this theorem is the following. We only need to worry about situations when the system gets very close to one of the axes. We focus on the case where p_1 is very small. In that scenario the effect of the type 1 species on p_2^k is negligible, meaning that type 2 evolves as if it were alone. On the other hand, for p_1 small, the total growth of type 1 after one iteration will be roughly ϕ_1 times a factor smaller than 1 corresponding to the competition effect coming from p_2. What condition (II.13) states is that, on average, this competition effect coming from type 2 (represented by the square root factor) is not strong enough to compensate the growth produced by ϕ_1, allowing thus p_1 to move away from low density values.
Tools: dynamical system observables, compactness, finite intersection property.

Remark II.2.3.2
In (II.13), the parameter ϕ_1 needs to grow roughly as $\sqrt{\phi_2 \log(\phi_2)}$ as a function of ϕ_2 in order for the left hand side to stay above 1. To see this, use the definition of κ_2 to write $\phi_2 = \frac{\kappa_2}{g_{\alpha(2)}(1-e^{-\kappa_2})} = \frac{\kappa_2}{(1-e^{-\kappa_2})G_{\alpha(2)}(1-e^{-\kappa_2})^3} = \frac{\kappa_2(1+\sqrt{1-4(1-\alpha(2))e^{-\kappa_2}(1-e^{-\kappa_2})})^3}{(1-e^{-\kappa_2})8e^{-3\kappa_2}}$, which says that κ_2 grows roughly as $\log(\phi_2)$, and then substitute this approximation in (II.13).

The next result states the domination counterpart to Theorem II.2.3.1.

Theorem II.2.3.3: Domination
Let $a_1(x)$ be the solution of $a_1(x) = x(1 - e^{-a_1(x)})$ and assume that ϕ_1 and ϕ_2 satisfy

$$a_1(\phi_1) \ < \ \frac{\phi_2}{1-\alpha(2)} \min\Big\{ g_{\alpha(2)}(1 - e^{-\frac{\phi_2}{2}}), g_{\alpha(2)}(1 - e^{-a_1(\phi_1)}) \Big\}. \tag{II.14}$$

Then for any initial condition p with $p_2 \in (0,1)$ we have

$$p_1^k \xrightarrow[k\to\infty]{} 0 \qquad \text{and} \qquad \liminf_{k\to\infty} p_2^k > 0.$$

Key idea.

Even though the condition given in (II.14) is again relatively obscure (see Figure II.8 for an approximation of the associated region), the basic idea behind this result is simple. Starting from any initial condition we show that the orbit of the dynamical system eventually reaches a set B where p_1 decays exponentially. We then make use of (II.14) to show that neither low nor high values of p_2 can take the dynamical system out of B, making it a "trapping" set where type 1 species dies out.
Tools: dynamical systems, attraction of fixed points, basin of attraction.

Remark II.2.3.4
It is easy to see that $a_1(\phi_1)$ is increasing with respect to ϕ_1, with $a_1(1) = 0$, so for a given ϕ_2, any value of ϕ_1 sufficiently close to 1 satisfies (II.14).

Simulations suggest that if ϕ_2 is smaller than but sufficiently close to $2\log 2$ and $\alpha(1), \alpha(2) \in (0,1)$, then there exists ϕ_1 smaller than ϕ_2 such that coexistence holds. See Figure II.7 (right) for a simulation which exhibits this behavior (note that both species have positive density to the left of the leftmost vertical line).

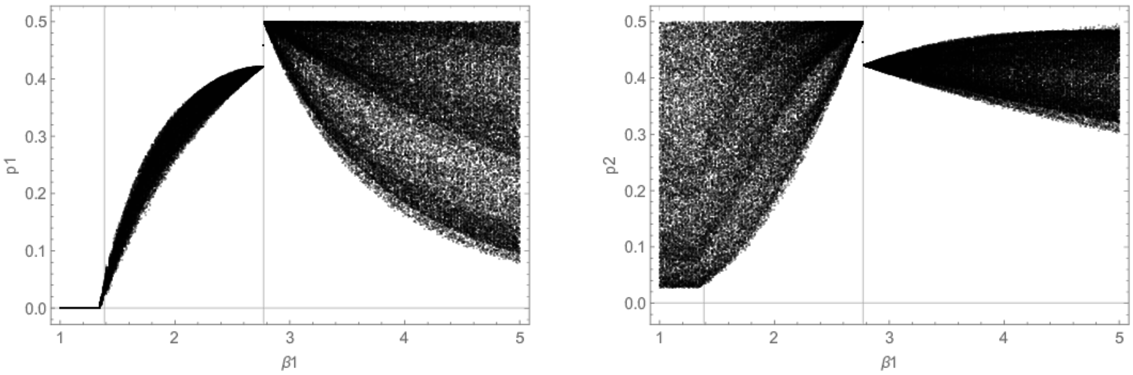


Figure II.6 – Bifurcation diagrams in $\beta(1)$ for type 1 on the left and type 2 on the right, with $\beta(2) = 4\log(2)$ and $\alpha(1) = \alpha(2) = 0$. From left to right, in each figure, the first vertical line is at $\phi_1 = 2\log 2$ and the second one at $\phi_1 = \phi_2$. These diagrams reflect theorems II.2.3.3(b) and theorem II.2.3.1. These diagrams depict regions corresponding to Theorems II.2.3.3 (dominance of type 2 over type 1) and II.2.3.1 (coexistence); coexistence corresponds to the region between the two vertical lines in both figures.

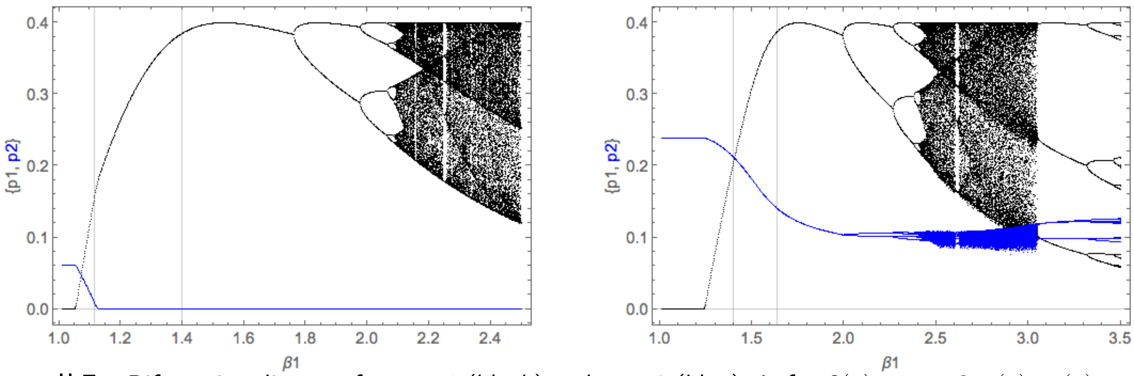


Figure II.7 – Bifurcation diagram for type 1 (black) and type 2 (blue). Left: $\beta(2) = 1.99\log(2)$, $\alpha(1) = 0.01$ and $\alpha(2) = 0.2$. From left to right, the first vertical line is at $\phi_1 = \phi_2$ while the second one is at $\phi_1 = 2\log 2$. Right: $\beta(2) = 2.6\log(2)$, $\alpha(1) = 0.01$ and $\alpha(2) = 0.1$. From left to right, the first vertical line is at $\phi_1 = 2\log 2$ while the second one is at $\phi_1 = \phi_2$.

Coexistence and domination in the MMM

We arrive finally at the main results of the chapter, which explore the possibility of domination and coexistence for the MMM. This is done by using the approximation theorem to transfer the properties of DS(h) derived in the last section to the associated particle systems for suitable families of parameters.

Let us stress again that, if we consider the MMM without epidemics, then the resulting process is nothing more than a multi-type contact process, for which it is known that the species with larger offspring parameter will always outcompete the other one (this has been proved for other choices of G_N, e.g. the result of [94] mentioned in Remark II.1.2.2, but in the current setting of mean-field growth it would be simple to prove). The upcoming results will show that, as advertised, there are choices of parameters for which there is coexistence even when one species has a larger offspring parameter, and hence that the introduction of forest fire dynamics can indeed lead to coexistence in a system which would otherwise show domination.

Let

$$\tau_N^i = \inf\{k \geq 1 \colon \eta_k^N(x) \neq i \ \forall x \in G_N\} = \inf\{k \geq 1 : \rho_k^{N,(i)} = 0\}$$

denote the extinction time of the type i.

Theorem II.2.3.5

Consider the two-species mean-field MMM running on a random 3-regular graph G_N. Suppose that for each N the initial density of the process ρ_0^N is in $(0,1)^2$, and that the sequence $\vec{\alpha}_N$ satisfies the conditions in Theorem II.2.1.3. Then there are constants $C = C(\rho_0^N) > 0$ and $\gamma \in (0,1)$ such that for $\underline{\alpha} = \min\{\alpha(1), \alpha(2)\}$ we have:

(1) (Coexistence) If $\vec{\alpha}_N$ and $\vec{\beta}$ satisfy the conditions of Theorem II.2.3.1, then

$$\mathbb{P}(\tau_N^1, \tau_N^2 \geq n) \ \geq \ \left(1 - C\theta_{\underline{\alpha}}(N)\right)^n . \tag{II.15}$$

(2) (Domination of type 2 over type 1) If $\vec{\alpha}_N$ and $\vec{\beta}$ satisfy the conditions of Theorem II.2.3.3, then

$$\mathbb{P}(\tau_N^2 \geq n) \ \geq \ \left(1 - C\theta_{\underline{\alpha}}(N)\right)^n \tag{II.16}$$

and

$$\mathbb{P}(\tau_N^1 \geq n) \ \leq \ 2 - (1-\gamma^n)^N - \left(1 - C\theta_{\underline{\alpha}}(N)\right)^n . \tag{II.17}$$

In particular, if we assume that $\rho_0^N \to p \in (0,1)^2$ as $N \to \infty$, then:

(1 ') For $\vec{\alpha}$ and $\vec{\beta}$ satisfying the conditions of Theorem II.2.3.1 and all $\varepsilon > 0$,

$$\mathbb{P}\left(\tau_N^1, \tau_N^2 \geq 1/\theta_{\underline{\alpha}}(N)^{1-\varepsilon}\right) \xrightarrow{N\to\infty} 1. \tag{II.18}$$

(2 ') For $\vec{\alpha}$ and $\vec{\beta}$ satisfying the conditions of Theorem II.2.3.3, and for all $\varepsilon > 0$, there is a $C' > 0$ depending only on p such that

$$\mathbb{P}(\tau_N^1 \leq C' \ln N) \xrightarrow{N\to\infty} 1 \quad \text{and} \quad P\left(\tau_N^2 \geq 1/\theta_{\underline{\alpha}}(N)^{1-\varepsilon}\right) \xrightarrow{N\to\infty} 1. \tag{II.19}$$

Recalling that $\theta_{\underline{\alpha}}(N)^{1-\varepsilon}$ is of larger order than $\ln(N)$, this gives domination.

Key idea.

> We use Theorem II.2.1.3 to copy the behavior of the dynamical system.
> **Tools:** Galton-Watson processes.

As a consequence if condition (II.8) is satisfied, then (in the mean-field case) we have that:

(a) Under the conditions of Theorem II.2.3.1 there is coexistence, in the sense that with high probability both species are present in the system for an amount of time of order at least $\theta_{\underline{\alpha}}(N)^{-1}$.

(b) Under the conditions of Theorem II.2.3.3 there is domination, in the sense that, with high probability, the extinction time of type 1 is at most of order N while type 2 survives for at least an amount of time of order at least $\theta_{\underline{\alpha}}(N)^{-1}$.

(c) The possibilities for survival and extinction listed in Section II.1.3 hold for the MMM.

(d) In particular, there exist $\phi_2 > 2\log(2)$ and $\phi_1 < \phi_1' < \phi_2$, such that in the MMM associated to (ϕ_1, ϕ_2) type 2 dominates over type 1 while the MMM associated to (ϕ_1', ϕ_2) is in the coexistence regime. This can be achieved, moreover, when $\alpha(1) = \alpha(2) = 0$.

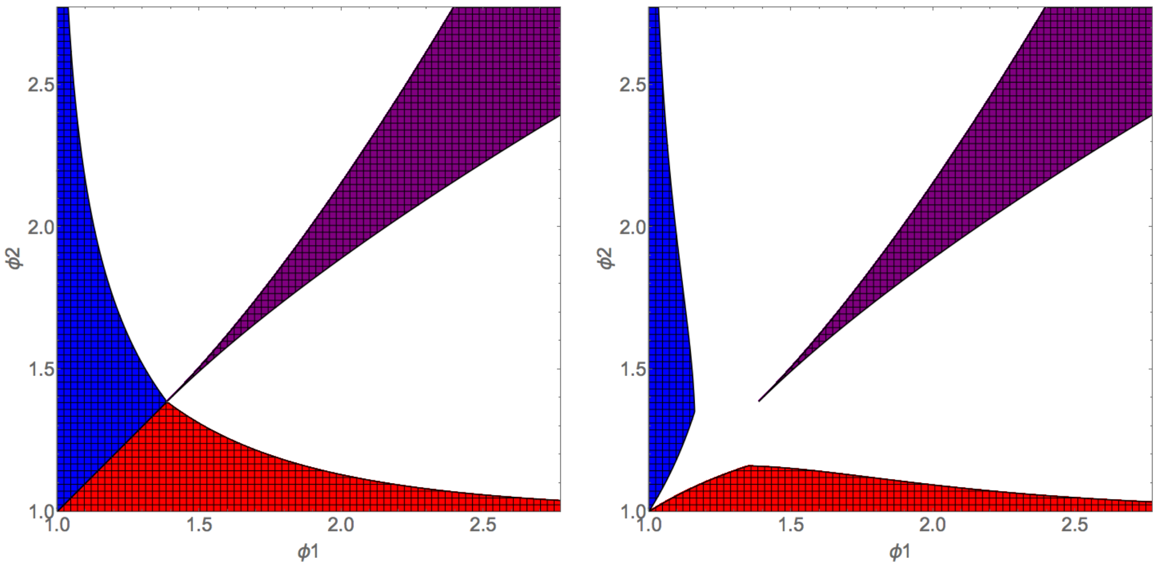


Figure II.8 – Summary of the domination and coexistence regimes for the MMM, for $\alpha(1) = \alpha(2) = 0$ on the left and $\alpha(1) = \alpha(2) = 0.1$ on the right. The white (resp. black) regions represent the domination regime of type 1 over type 2 (resp. type 2 over type 1); these regions are justified by Theorems II.2.3.3 and II.2.3.5. The gray regions roughly correspond the coexistence regime, and are justified by Theorems II.2.3.1 and II.2.3.5 (the coexistence regions are only approximate in the sense that they where plotted based on their asymptotic behavior: as $\phi_2 \to \infty$, ϕ_1 grows as $\sqrt{\phi_2 \log(\phi_2)}$, see Remark II.2.3.2). The behavior of the white regions is not determined by our results.

Figure II.8 contains a sketch of the regions of the phase diagram of the process which have been probed in Theorem II.2.3.5, which in particular makes the existence of the parameter triplets $(\phi_1, \phi_1', \phi_2)$ referred to in (d) above apparent. In fact, as $\phi_2 \to \infty$ we have that ϕ_1' is of order $\sqrt{\phi_2 \log(\phi_2)}$ (see II.2.3.2 for an explanation), and hence we can find $\phi_1' < \phi_2$ satisfying the corollary for $\beta(2)$ sufficiently large.

II.3 Proofs of the convergence and approximation results

We begin with the simple proof of the formula for g_α.

Proof of Proposition II.2.1.1. Recall that $\mathcal{T}$ denotes an infinite 3-tree, $\mathbf{P}_p$ denotes the site percolation measure on $\mathcal{T}$ with density p, and $\mathcal{C}_r$ denotes the percolation cluster containing a given vertex r. The cases $\alpha = 0$ and $\alpha = 1$ are straightforward, so we turn to the case $\alpha \in (0,1)$, where we have

$$\mathbf{E}_p((1-\alpha)^{|\mathcal{C}_r|}\mathbb{1}_{|\mathcal{C}_r|>0}) = \sum_{n=1}^{\infty}(1-\alpha)^n \mathbf{P}_p(|\mathcal{C}_r| = n).$$

Let A_n be the number of possible connected components of size n in a 3-tree rooted at r, so that $\mathbf{P}_p(|\mathcal{C}_r| = n) = A_n p^n (1-p)^{n+2}$ (notice that $n+2$ is the number of vacant sites surrounding a cluster $\mathcal{C}_r$ of size n). Noting that a 3-tree is a root connected to three binary trees and recalling that the analog of A_n for a binary tree is given by the Catalan numbers C_n, we get

$$A_0 = 1 \quad \text{and} \quad A_{n+1} = \textstyle\sum_{i=0}^{n}\sum_{j=0}^{n-i} C_i C_j C_{n-i-j}. \tag{II.20}$$

Defining the generating functions $A(x) = \sum_{n=0}^{\infty} A_n x^n$ and $C(x) = \sum_{n=0}^{\infty} C_n x^n$, the above equation gives

$$A(x) = xC(x)^3 + 1 = x\Big(\tfrac{1-\sqrt{1-4x}}{2x}\Big)^3 + 1,$$

where we have used the explicit formula for $C(x)$ (see [104]). Using this above yields

$$\begin{aligned}\mathbf{E}_p((1-\alpha)^{|\mathcal{C}_r|}\mathbb{1}_{|\mathcal{C}_r|>0}) &= \sum_{n=1}^{\infty}(1-\alpha)^n p^n (1-p)^{n+2} A_n = (1-p)^2\big(A((1-\alpha)p(1-p)) - 1\big) \\ &= \frac{\Big(1-\sqrt{1-4(1-\alpha)p(1-p)}\Big)^3}{8(1-\alpha)^2 p^2}.\end{aligned}$$

□

The proof of Theorem II.2.1.2 is an adaptation of the proof of [42, Theorem 4] for the one-species model running on the torus, so we will only explain what needs to be changed. The extension to $m > 1$ is relatively straightforward, so we will focus first on the adaptations needed to drop the $\alpha_N \to 0$ assumption. The main ingredient in their proof consists in considering *bad* sites, which are sites x such that the density of occupied sites in the ball of radius r_N around it is far from the global density of occupied sites, and then proving (see their Prop. 5.1) that starting one iteration of the system with a small enough density of bad points yields a small density of bad points after one time step.

We introduce the following definitions:

$$B(x,r) := \{y \in R_N : d(x,y) \le r\}, \tag{II.21}$$
$$V(r) := |B(x,m)| = 3\cdot 2^r - 2, \tag{II.22}$$
$$d_k^{N,(i)}(x) := \frac{1}{V(r_N)} \sum_{y\in B(x,r_N)} \mathbb{1}_{\{\eta_k(y)=i\}}, \tag{II.23}$$
$$G_k^N(\varepsilon) := \{x \in G_N : \textstyle\sum_{i=1}^m |d_k^{N,(i)}(x) - h_i^k(p)| < \varepsilon\}. \tag{II.24}$$

In the next lemma we will use the same random variables defined in the proof of [42, Thm. 4], only changing their C_0 by $\mathcal{C}_r$.

Lemma II.3.0.1
Assume $m = 1$. Given $\varepsilon > 0$ there exists N sufficiently large such that

$$\mathbb{E}\big(|\tilde{\rho}_{k+1}^N - \hat{\rho}_{k+1}^N|\big) \le \varepsilon.$$

Proof. We will use $\delta_1 > 0$, $\delta_2 > 0$ as small as needed. Changing the proof of convergence given for $|d_k^N(x) - d_k^N(0)|$ in [42, Lem. 5.4] by

$$\begin{aligned}&\mathbb{P}\left(|d_k^N(i) - d_k^N(0)| > \delta_1 \text{ for some } x \in B(0,l_N)\right) \\ &\quad\le V(l_N)\sup_{x\in B(0,l_N)} \mathbb{P}\big(|d_k^N(x) - d_k^N(0)| > \delta_1\big) \\ &\quad= V(l_N)\sup_{x\in B(0,l_N)} \mathbb{P}\Big(\Big|\tfrac{1}{V(r_N)}\Big(\textstyle\sum_{y\in B(x,r_N)\setminus B(0,r_N)} \eta_k^N(y) - \sum_{y\in B(0,r_N)\setminus B(x,r_N)} \eta_k^N(y)\Big)\Big| > \delta_1\Big) \\ &\quad\le \sup_{x\in B(0,l_N)} \frac{V(l_N)\mathsf{Var}\Big(\Big|\sum_{y\in B(x,r_N)\setminus B(0,r_N)} \eta_k^N(y) - \sum_{y\in B(0,r_N)\setminus B(x,r_N)} \eta_k^N(y)\Big|\Big)}{V(r_N)^2\delta_1^2} \\ &\quad\le \sup_{x\in B(0,l_N)} \frac{V(l_N)\mathsf{Var}\Big(\sum_{y\in B(x,r_N)\setminus B(0,r_N)\cup B(0,r_N)\setminus B(x,r_N)} \eta_k^N(y)\Big)}{V(r_N)^2\delta_1^2}\end{aligned}$$

$$\leq \frac{2V(l_N)V(r_N)\mathrm{Var}\big(\eta_k^N(0)\big)}{V(r_N)^2\delta_1^2}$$

gives

$$\mathbb{P}\left(|d_k^N(i)-d_k^N(0)|>\delta_1 \text{ for some } i\in B(0,l_N)\right)\leq \frac{2V(l_N)}{V(r_N)\delta_1^2}\leq \frac{1}{\delta_1^2}2^{l_N-r_N+2}\longrightarrow 0.$$

Now define $Y(\delta)=\#\left(\xi_{1/2}^{h_k(p)+2\delta_1}\setminus\xi_{1/2}^{h_k(p)}\right)$. For the inequality in (5.4) in [42] in our case we consider the following bound

$$\begin{aligned}\mathbb{P}&\left(\xi_1^{h_k(p)+2\delta_1,N}(0)=0,\xi_1^{h_k(p),N}(0)=1,\#\xi_{1/2}^{h_k(p)+2\delta_1}<\infty\right)\\&\leq\mathbb{E}\left(1-\left(\tfrac{(1-\alpha)}{2}\right)^{Y(\delta_1)}\mathbb{1}_{\{\#\xi_{1/2}^{h_k(p)+2\delta_1}<\infty\}}\right)\\&\leq\textstyle\sum_{i=0}^{\infty}\left(1-\left(\tfrac{(1-\alpha)}{2}\right)^i\right)\mathbb{P}\left(Y(\delta_1)=i\middle|\#\xi_{1/2}^{h_k(p)+2\delta_1}<\infty\right).\end{aligned}$$

The last term converges to 0 when $\delta_1\to 0$, because (here A_j comes from (II.20))

$$\begin{aligned}\mathbb{P}&\left(\xi_{1/2}^{h_k(p)+2\delta_1}=\xi_{1/2}^{h_k(p)}\middle|\#\xi_{1/2}^{h_k(p)+2\delta_1}<\infty\right)\\&=\frac{e^{-\beta\left(h_k(p)+2\delta_1\right)}+\sum_{j=1}^{\infty}A_j\left(1-e^{-\beta h_k(p)}\right)^je^{-(j+2)\beta\left(h_k(p)+2\delta_1\right)}}{1-\mathbb{P}_q(|\mathcal{C}_r|=\infty)}\xrightarrow[\delta_1\to 0]{}1,\end{aligned}$$

where $q=1-e^{-\beta\left(h_k(p)+2\delta_1\right)}$. The last limit is obtained using the proof of Proposition II.2.1.1 and the Dominated Convergence Theorem. The conclusion follows as in [42]. □

Proof of Theorem II.2.1.2. The case $m=1$ follows by changing Lemma 5.4 in [42, Theorem 4] by Lemma II.3.0.1 above. The proof for the case $m\geq 2$ is just an adaptation of the case $m=1$ multiple species, here we show the key points. In these adaptations one should always use $\|\cdot\|_1$ instead of $|\cdot|$. Recall that the evolution in the growth step of a given site x depends on the local density $d_k^{N,(i)}$. Given that each occupied site x of type i sends a Poisson$[\beta(i)/V(r_N)]$ number of births to each of its $V(r_N)$ neighbors in $B(x,r_N)$ it follows that each site receives a Poisson$[\beta(i)d_k^{N,(i)}]$ number of births of type i and a total Poisson$[\sum_{i=1}^m\beta(i)d_k^{N,(i)}]$ number of births. Then, given $\eta_k^N(x)$, the site x has a particle of type i after the growing stage with probability

$$\mathbb{P}\left(\eta_{k+1/2}^N(x)=i\right)=\left(1-\exp\left(\textstyle\sum_{j=1}^m\beta(j)d_k^{N,(j)}\right)\right)\frac{\beta(i)d_k^{N,(i)}}{\sum_{j=1}^m\beta(j)d_k^{N,(j)}}.$$

The random variables $\tilde{\eta}_k^N$, η_k^N and $\hat{\eta}_k^N$ have to be extended for multi-species and the coupling between these has to be reformulated accordingly. These are simple adaptations so they are left to the reader, together with the remainder of the proof. □

Proof of Theorem II.2.1.3. Start defining the event $H_N=\{x\in G_N: G_N\cap B(x,L_N)$ is a finite 3-tree$\}$ with $L_N=\log_2(N)/5$. Observe first that, since $\delta>0$ is arbitrary, and from the uniform continuity of h, we only need to prove the statement of the theorem for $k=1$. Even further, it is enough to show that for any fixed $j\in\{1,\dots,m\}$ and $\delta>0$ we can find C such that

$$\mathbb{P}\left(\left|\rho_1^{N,(j)}-h^j(\rho_0^N)\right|>\delta\right)\;\leq\;C\theta_{\alpha(j)}(N). \tag{II.25}$$

Fix then any such j and define $\tilde{\eta}_1^N$ as the modified process with $\tilde{\eta}_1^N(x)=j$ if the vertex x belongs to H_N and at time $\frac{1}{2}$ is occupied by an individual of type j that survives the epidemic when one ignores infections arising outside $B(x, L_N)$. Defining $\tilde{\rho}_1^N$ analogously to ρ_1^N as the density of $\tilde{\eta}_1^N$, we can bound the event under the probability (II.25) by

$$\mathsf{Q}_N := \Big\{ \big|\rho_1^{N,(j)} - \tfrac{1}{N}|\eta_1^{N,(j)} \cap H_N|\big| + \big|\tfrac{1}{N}|\eta_1^{N,(j)} \cap H_N| - \tilde{\rho}_1^{N,(j)}\big| \\ + |\tilde{\rho}_1^{N,(j)} - \mathbb{E}(\tilde{\rho}_1^{N,(j)})| + |\mathbb{E}(\tilde{\rho}_1^{N,(j)}) - h^j(\rho_0^N)| > \delta \Big\},$$

and using Markov's inequality we obtain

$$\mathbb{P}(\mathsf{Q}_N) \leq \tfrac{4}{\delta}\mathbb{E}\left(\big|\rho_1^{N,(j)} - \tfrac{1}{N}|\eta_1^{N,(j)} \cap H_N|\big| \right) + \tfrac{4}{\delta}\mathbb{E}\left(\big|\tfrac{1}{N}|\eta_1^{N,(j)} \cap H_N| - \tilde{\rho}_1^{N,(j)}\big| \right) \\ + \tfrac{4}{\delta}\mathbb{E}\left(|\tilde{\rho}_1^{N,(j)} - \mathbb{E}(\tilde{\rho}_1^{N,(j)})| \right) + \mathbb{P}\left(|\mathbb{E}(\tilde{\rho}_1^{N,(j)}) - h^j(\rho_0^N)| > \tfrac{\delta}{4} \right), \quad \text{(II.26)}$$

so the result will follow by showing that each term in the expression above is bounded by $C\theta_{\alpha(j)}(N)$ for some C independent of ρ_0^N. For the first term on the right hand side we use the bound

$$\mathbb{E}\left(\big|\rho_1^{N,(j)} - \tfrac{1}{N}|\eta_1^{N,(j)} \cap H_N|\big| \right) \leq \frac{\mathbb{E}(G_N \setminus H_N)}{N}$$

where [42, Lem. 3.2] gives $\frac{\mathbb{E}(|G_N \setminus H_N|)}{N} \leq 4N^{-3/5}$, and for the third term in (II.26) we use

$$\mathbb{E}\left(|\tilde{\rho}_1^{N,(j)} - \mathbb{E}(\tilde{\rho}_1^{N,(j)})| \right) \leq \sqrt{\mathsf{Var}(\tilde{\rho}_1^{N,(j)})},$$

where independence between any pair of events of the form $x \in \tilde{\eta}_1^{N,(j)}$ and $y \in \tilde{\eta}_1^{N,(j)}$ for $x, y \in H_N$ with $d(x, y) > 2L_N$ gives

$$\mathsf{Var}(\tilde{\rho}_1^{N,(j)}) \leq N^{-2}\,|\{(x,y) \in H_N \times H_N,\ d(x,y) \leq 2L_N\}| \\ \leq N^{-2} \sum\nolimits_{x \in G_N} |B(x, L_N)| \ = \ N^{-2}(2N \cdot N^{2/5}) \ \leq \ 2N^{-3/5},$$

so for both terms we obtain the bound $\frac{16}{\delta}N^{-3/10}$ which is by definition smaller than $C\theta_{\alpha(j)}(N)$. To control the second and fourth terms in (II.26) observe that by translation invariance we can fix any vertex $r \in G_N$ and use the definition of $\tilde{\eta}_1^N$ to express $\mathbb{E}(\tilde{\rho}_1^{N,(j)})$ as

$$\mathbb{E}(\tilde{\rho}_1^{N,(j)}) \ = \ \mathbb{P}(\tilde{\eta}_1^N(r) = j) \ = \ \mathbb{E}\Big(\mathbb{1}_{\{r \in \eta_{1/2}^{N,(j)} \cap H_N\}} (1 - \alpha_N(j))^{|\mathcal{C}_r^j \cap B(r, L_N)|} \Big),$$

where $\mathcal{C}_r^j$ is the connected component of the type j containing r. Now, the event $r \in H_N$ implies that $B(r, L_N)$ is a 3-tree, and by the mean-field assumption for the growth stage, at time $1/2$ each vertex is occupied by an individual of the j type independently with probability $q = f_{\vec{\beta}}^{(j)}(\rho_0^N)$. As a result, $|\mathcal{C}_r^j \cap B(r, L_N)|$ will be the size of the cluster containing r in the percolated 3-tree, which we represent as the total amount of individuals of a Galton-Watson process $Z_0, Z_1, \ldots, Z_{L_N - 1}$. More precisely since a 3-tree can be seen as a vertex connected to the root of three binary tress, we set the offspring distribution on the first generation of the Galton-Watson process as Binomial$[3, q]$, while at all subsequent generations it is Binomial$[2, q]$, with $Z_0 = \mathbb{1}_{\{r \in \eta_{1/2}^{N,(j)}\}}$, giving the expression

$$\mathbb{E}(\tilde{\rho}_1^{N,(j)}) = \ \mathbb{E}\big(\mathbf{1}_{\{r \in H_N\}} Z_0 (1 - \alpha_N(j))^{Z_0 + Z_1 + \cdots + Z_{L_N - 1}} \big) \\ = \ \mathbb{P}(r \in H_N)\, \mathbb{E}\big(Z_0 (1 - \alpha_N(j))^{Z_0 + Z_1 + \cdots + Z_{L_N - 1}} \big),$$

where the second equality comes from the fact that given the event $r \in H_N$, the variables $Z_0, Z_1, \ldots, Z_{L_N-1}$ do not depend on the particular realization of G_N.

The next result, whose proof we postpone to the appendix, allows us to control the expectation $\mathbb{E}(\tilde{\rho}_1^{N,(j)})$:

Lemma II.3.0.2
Take a sequence $(\alpha_N)_{N\geq 0} \subseteq [0,1]$ converging to some α, and a Galton-Watson process $Z_0, Z_1, \ldots$ as above. If the condition

$$-\alpha_N \log N / \log \alpha_N \longrightarrow \infty$$

is satisfied, then there is $C > 0$ independent of q such that for all N,

$$\left|\mathbb{E}\big(Z_0(1-\alpha_N)^{Z_0+Z_1+\cdots+Z_{L_N-1}}\big) - g_{\alpha_N}(q)\right| \leq C\theta_\alpha(N). \tag{II.27}$$

The same bound holds for $\mathbb{E}\big(Z_0(1-\alpha_N)^{Z_0+Z_1+\cdots+Z_{L_N-1}}\mathbb{1}_{\{Z_{L_N-1}=0\}}\big)$.

Using Lemma II.3.0.2, the uniform convergence of $g_{\alpha_N(j)}$ to $g_{\alpha(j)}$, and that $\mathbb{P}(0 \in H_N) \to 1$, we deduce that there is N_0 independent of ρ_0^N such that $|\mathbb{E}(\rho_1^{N,(j)}) - h^j(\rho_0^N)| < \delta/4$ for all $N \geq N_0$. In particular, we deduce

$$\mathbb{P}\big(|\mathbb{E}(\rho_1^{N,(j)}) - h^j(\rho_0^N)| > \tfrac{\delta}{4}\big) \;\leq\; C\theta_{\alpha(j)}(N)$$

for some $C > 0$ independent of ρ_0^N, so it only remains to control the second term in (II.26). Notice that $\tilde{\rho}_1^{N,(j)} - \frac{1}{N}|\eta_1^{N,(j)} \cap H_N|$ corresponds by definition to the fraction of vertices x in H_N which at time $\frac{1}{2}$ are occupied by an individual of type j that survives the restricted epidemic but not the unrestricted one. In particular, for any such vertex there must be an open path to the boundary of $B(x, L_N)$ used by the unrestricted infection to kill x, so we deduce

$$\begin{aligned}
\mathbb{E}\Big(\Big|\tfrac{1}{N}|\eta_1^{N,(j)} \cap H_N| - \tilde{\rho}_1^{N,(j)}\Big|\Big) &\leq \mathbb{E}\big(\mathbb{1}_{\{r\in\eta_{1/2}^{N,(j)}\cap H_N\}}(1-\alpha_N(j))^{|\mathcal{C}_r^j\cap B(r,L_N)|}\mathbb{1}_{\{\mathcal{C}_r^j\not\subseteq B(r,L_N)\}}\big)\\
&\leq \mathbb{E}\big(Z_0(1-\alpha_N(j))^{Z_0+Z_1+\cdots+Z_{L_N-1}}\mathbb{1}_{\{Z_{L_N-1}>0\}}\big),
\end{aligned}$$

where the variables $Z_0, \ldots, Z_{L_N-1}$ are defined as before. This last bound is equal to

$$\mathbb{E}\big(Z_0(1-\alpha_N(j))^{Z_0+Z_1+\cdots+Z_{L_N-1}}\big) - \mathbb{E}\big(Z_0(1-\alpha_N(j))^{Z_0+Z_1+\cdots+Z_{L_N-1}}\mathbb{1}_{\{Z_{L_N-1}=0\}}\big),$$

but from Lemma II.3.0.2 both therms are at distance at most $C\theta_\alpha(N)$ from $g_{\alpha_N(j)}(q)$, so

$$\mathbb{E}\Big(\Big|\tfrac{1}{N}|\eta_1^{N,(j)} \cap H_N| - \tilde{\rho}_1^{N,(j)}\Big|\Big) \;\leq\; 2C\theta_\alpha(N),$$

giving the result. □

II.4 Proofs for the one-type model

Proof of Theorem II.2.2.2(i). We start by sampling the graph and fixing a labeling on it. We will couple the MM process after the growing stage, i.e. $(\eta_{n+1/2}^N)_{n\in\mathbb{N}}$, with a Galton-Watson process $(Z_n)_{n\in\mathbb{N}}$. To this end we consider stacks $\{(\mathrm{O}_i^j)_{i\in\mathbb{N}}\}_{j\in\mathbb{N}}$ of i.i.d. random variables distributed according to the offspring distribution of the MM model and stacks $\{(\mathsf{E}_i^j)_{i\in\mathbb{N}}\}_{j\in\mathbb{N}}$ of i.i.d. random variables with $\mathbb{P}(\mathsf{E}_i^j = 1) = 1 - \mathbb{P}(\mathsf{E}_i^j = 0) = \alpha$. We use these random variables to define the MM process $(\eta_n)_{n\in\mathbb{N}}$ on G_N with a given initial configuration η_0 in the obvious way, using O_i^j and E_i^j to determine the offspring (if occupied) and an epidemic event at site i and time j. Next we define the Galton-Watson branching process $(Z_n)_{n\in\mathbb{N}}$. It starts with $Z_0 = |\eta_0|$ and uses the random variable $\mathsf{X}_i^j := \mathsf{O}_i^j(1-\mathsf{E}_i^j)$ to determine the offspring of i-th individual of Z_{j-1}. It is clear that $|\eta_n| \le Z_n$ for all n (we omit the simple argument).

Let now $\tau_k^{\mathrm{GW}} = \inf\{n \ge 1 : Z_n = 0\}$ be the extinction time of the Galton-Watson process started with k particles. From branching process theory we know that, since the mean offspring is ϕ_N, then starting with 1 particle we have that

$$\mathbb{P}(\tau_1^{\mathrm{GW}} \ge n) \le \begin{cases} (\phi_N)^n & \text{if } \phi_N < 1, \\ \frac{2}{\mathsf{Var}(\mathsf{X}_1^1)n} & \text{if } \phi_N = 1. \end{cases}$$

Focusing on the case $\phi_N < 1$, it follows that, since $|\eta_0| \le N$, there exists a $c > 0$ such that

$$\begin{aligned}\mathbb{P}(\tau_N \ge n) \le \mathbb{P}(\tau_{|\eta_0|}^{\mathrm{GW}} \ge n) &= \mathbb{P}\left(\max_{i\in\{1,2,\dots,|\eta_0|\}} \tau_1^{\mathrm{GW}}(i) \ge n\right) \\ &\le 1-(1-(\phi_N)^n)^{|\eta_0|} \le 1-(1-(\phi_N)^n)^N \le 1-\exp\left(-c(\phi_N)^n N\right).\end{aligned}$$

The next-to-last bound is what we wanted. The last bound yields the estimated on the expectation: in fact, for $K_0 = \log_{\phi^{-1}}(N)$ there exists $C > 0$, such that

$$\textstyle\sum_{n\in\mathbb{N}} (1-\exp(-c(\phi_N)^n N)) \le K_0 + \sum_{n\ge K_0} (1-\exp(-c(\phi_N)^n N)) \le K_0 + \frac{c}{1-\phi_N} \le C\log(N).$$

The same arguments yield the result in the case $\phi_N = 1$. ☐

The proof of Theorem II.2.2.2(ii) will be adapted from that of Theorem II.2.2.3, so we turn to that proof next.

Proof of Theorem II.2.2.3. We will prove that there is a $c \in (0,1)$ depending only on ρ_0 and α_N satisfying

$$\mathbb{P}(\tau_N \ge n) = (1-\exp(-cN))^n \quad \forall n \in \mathbb{N}.$$

The basic idea is to keep track of the isolated particles in each stage (growth and epidemic). Notice that since we are in the mean-field case we can suppose without loss of generality that we start from the product measure, simply because the growth step returns a product measure anyway. We will need an upper bound on the number of empty sites. We will say that a site is infected by the epidemic if the epidemic attacks the site, irrespective of whether the site is occupied or not. Let EP_i denote the number of sites infected by the epidemic at time i. Since the mean number of sites attacked by the epidemic is $\alpha_N N$, a Chernoff bound yields

$$\mathbb{P}\left(\left|\mathsf{EP}_i - \alpha_N N\right| \le \alpha_N N/2\right) \ge 1 - 2\exp(-\alpha_N^2 N/2). \tag{II.28}$$

In particular, uniformly on the initial condition and w.h.p. the following event occurs:

$$\mathsf{D}_i := \{\rho_i \le 1 - \alpha_N/2\}. \tag{II.29}$$

Now we explore percolation properties of the graph, since our process after the growth stage is a site-percolation with parameter $f_\beta(p)$ (when we start with density p). Denote by $G_N(p)$ the induced subgraph given by the open sites in percolation of parameter p. For every vertex u in the graph let $\mathsf{X}_u^i = \mathbb{1}_{\{u \text{ isolated in } G_N(p) \text{ at time i+1/2}\}}$. The family $\{\mathsf{X}_u^i\}_{u \in G_N}$ is not independent, in fact $\mathsf{X}_u^i = 1$ iff the neighbors of u are closed in the percolation. In order to produce an independent family we will divide the graph into a set of disjoint claws, by which we mean one vertex joined with its three neighbors.

By hypothesis the graph has a 3-independent set I_N of size at least $b_1 N$, which implies that it has at least $b_1 N$ disjoint claws (each given by a vertex in I_N together with its three neighbors).

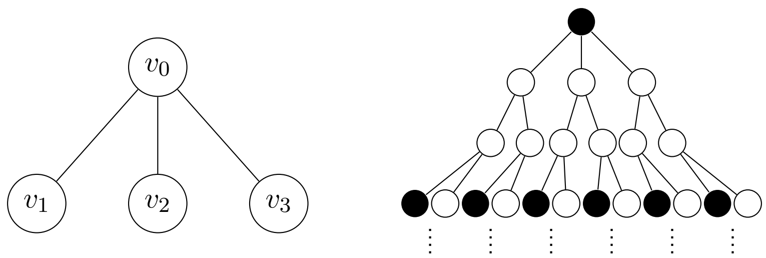


Figure II.9 – The left figure represents a claw, also called a cherry or $K_{1,3}$ in the literature. The right figure represents the local behavior of a 3-independent set on the 3-regular tree (black vertices belong to I_N).

Consider the family $(\mathsf{X}_u^i)_{u \in I}$, and notice that it is made of independent random variables.

Let $\mathsf{p}_{i+1/2}$ be the density of isolated particles after the growing stage in the i-th iteration of the system. Let also p_{i+1} be the density of isolated sites that survive the epidemic:

$$\mathsf{p}_{i+1} = \frac{1}{N} \sum\nolimits_{v \in G_N} \eta_{i+1}(v) \mathsf{X}_v^i$$

Let $\rho_{I_N} = |I_N|/N$ and define $\mathsf{Iso}_N(\rho) = (1 - e^{-\beta\rho}) e^{-3\beta\rho} \rho_{I_N}$; this quantity will be important in the developments that follow as it represents the expected number of isolated particles after the growth stage in I. More formally, starting with density ρ, the probability that a given site is empty after the growth stage is by translation invariance equal to the expectation of the density after the growth stage starting with density ρ, and this expectation is equal to $(1 - \beta/N)^{\rho N} \approx e^{\beta\rho}$. For simplicity we will use the function Iso_N not just as an approximation for the expectation, but instead of the actual expectation; this may be justified from the fact that this approximation has a rate of convergence which is much faster than the approximations which we do in what follows, but we leave these details to the reader. Notice that we can suppose that we sample the graph, we choose I_N, and then we run the process, so that the function Iso at this point is deterministic; we omit the dependence on the graph and on N. Define $\mathsf{Iso}(\rho) := (1 - e^{-\beta\rho}) e^{-3\beta\rho} b_1$ and notice that $\mathsf{Iso}_N(x) \geq \mathsf{Iso}(x)$ for all $x \in [0, 1]$, since we are conditioning on $G_N \in \mathcal{R}_N(b_1)$. Let

$$m = m(\delta) = \min_{x \in [\delta, 1-\delta]} \mathsf{Iso}(x).$$

This minimum is attained at one of the boundary values, as can be checked from simple properties of $\mathsf{Iso}(x)$. In particular, $\mathsf{Iso}'(x)$ vanishes at $x = \bar{x}$ defined as the unique point satisfying $\exp(\beta\bar{x}) = 4/3$, which is a maximum. The values of the function are strictly positive inside $[0, 1]$, hence $m > 0$. Also 0 is not attractive since the derivative there is $b_1\beta > 1$.

Lemma II.4.0.1
For small enough $\varepsilon > 0$ there exists $\bar{\delta} \in (0, 1/2)$ such that for every $\delta \in (0, \bar{\delta})$ satisfying $m(\delta)(1-\alpha_N)(1-2\varepsilon) \leq x \leq 1-\alpha_N/2$, we have $m(\delta) \leq \mathsf{Iso}(x)$.

Proof. Notice that $\lim_{x\to 0} \frac{x}{\mathsf{Iso}(x)} = \frac{1}{b_1\beta}$, so for $\gamma \geq 0$ small, there exists θ_γ such that if $0 \leq x \leq \theta_\gamma$, then

$$\frac{x(1-\alpha_N)(1-2\varepsilon)}{\mathsf{Iso}(x(1-\alpha_N)(1-2\varepsilon))} \leq \frac{(1+\gamma)}{b_1\beta} \iff x(1-\alpha_N)\beta b_1 \frac{(1-2\varepsilon)}{(1+\gamma)} \leq \mathsf{Iso}(x(1-\alpha_N)(1-2\varepsilon)).$$

But since $1 < (1-\alpha_N)\beta b_1$, if we choose γ and ε such that

$$(1-\alpha_N)\beta b_1 \frac{(1-2\varepsilon)}{(1+\gamma)} \geq 1$$

then we have the conclusion, since it is enough to consider $\bar{\delta}$ such that $m \leq \theta_\gamma$. □

Fix some small $\varepsilon > 0$ and choose δ such that $m(\delta)(1-\alpha_N)(1-2\varepsilon) \leq \rho_0^N$; this can be done because $m(\delta)$ is decreasing in δ. The event $\{\rho_0^N \leq 1-\alpha_N/2\}$, on the other hand, happens with high probability thanks to (II.28)/(II.29). Hence with our parameter choices we may apply Lemma II.4.0.1. Now we have all the elements to prove the theorem. Define the events

$$\begin{aligned}
\mathsf{A}_i &= \{m(1-\alpha_N)(1-2\varepsilon) \leq \rho_i \leq 1-\alpha_N/2\}\\
\mathcal{A}_i &= \cap_{j=0}^{i} \mathsf{A}_j\\
\mathsf{B}_{i+1/2} &= \{|\mathsf{p}_{i+1/2} - \mathsf{Iso}_N(\rho_i)| \leq \varepsilon \mathsf{Iso}_N(\rho_i)\}\\
\mathsf{C}_{i+1} &= \{|\mathsf{p}_{i+1} - \mathsf{p}_{i+1/2}(1-\alpha_N)| \leq \varepsilon(1-\varepsilon)(1-\alpha_N)\mathsf{p}_{i+1/2}\}\\
\mathsf{E}_i &= \{\rho_i \geq m(1-\alpha_N)(1-2\varepsilon)\}
\end{aligned}$$

Observe that, from the previous comment and (II.28), $\mathbb{P}(\mathsf{A}_0) > 1-\exp(-\alpha_N^2 N/2)$ by choosing $\delta > 0$ sufficiently small. Also observe that $\mathbb{P}(\tau_N \geq n+1) \geq \mathbb{P}(\mathcal{A}_{n+1}) = \mathbb{P}(\mathsf{A}_0)\prod_{i=0}^{n} \mathbb{P}(\mathcal{A}_{i+1}|\mathcal{A}_i)$ and that we have the decomposition

$$\begin{aligned}
\mathbb{P}(\mathcal{A}_{i+1}|\mathcal{A}_i) &= \mathbb{P}(\mathsf{E}_{i+1}|\mathcal{A}_i, \mathsf{D}_{i+1})\mathbb{P}(\mathsf{D}_{i+1}|\mathcal{A}_i) \geq \mathbb{P}(\mathsf{C}_{i+1}, \mathsf{B}_{i+1/2}|\mathcal{A}_i, \mathsf{D}_{i+1})\mathbb{P}(\mathsf{D}_{i+1}|\mathcal{A}_i)\\
&= \mathbb{P}(\mathsf{C}_{i+1}|\mathcal{A}_i, \mathsf{B}_{i+1/2}, \mathsf{D}_i)\mathbb{P}(\mathsf{B}_{i+1/2}|\mathcal{A}_i, \mathsf{D}_{i+1})\mathbb{P}(\mathsf{D}_{i+1}|\mathcal{A}_i);
\end{aligned} \tag{II.30}$$

the inequality follows from $\rho_i \geq \mathsf{p}_i$ and noting that, on the events $\mathsf{B}_{i+1/2}$ and C_{i+1} we have

$$\begin{aligned}
|\mathsf{p}_{i+1} - \mathsf{Iso}_N(\rho_i)(1-\alpha_N)| &\leq \varepsilon \mathsf{p}_{i+1/2}(1-\alpha_N) + (1-\alpha_N)\varepsilon \mathsf{Iso}_N(\rho_i)\\
&\leq \varepsilon(1-\alpha_N)(1-\varepsilon)\mathsf{Iso}_N(\rho_i) + (1-\alpha_N)\varepsilon \mathsf{Iso}_N(\rho_i) \leq 2\varepsilon(1-\varepsilon)(1-\alpha_N)\mathsf{Iso}_N(\rho_i),
\end{aligned}$$

which together with Lemma II.4.0.1 gives $\rho_{i+1} \geq \mathsf{p}_{i+1} \geq \mathsf{Iso}_N(\rho_i)(1-\alpha_N)(1-2\varepsilon) \geq m(1-\alpha_N)(1-2\varepsilon)$ as needed. We need to bound the product in the second line of (II.30). The bound for the third factor is obtained from (II.28). The middle factor can be bounded using a Chernoff bound similarly to (II.28) (notice that independence is crucial here again),

$$\begin{aligned}
\mathbb{P}(\mathsf{B}_{i+1/2}|\mathcal{A}_i, \mathsf{D}_{i+1}) &= 1 - \mathbb{P}(\mathsf{B}_{i+1/2}^c|\mathcal{A}_i, \mathsf{D}_{i+1}) \geq 1 - \mathbb{E}(\mathbb{E}(2e^{-2N\mathsf{Iso}_N(\rho_i)^2\varepsilon^2}|\rho_i, \mathcal{A}_i, \mathsf{D}_{i+1})|\mathcal{A}_i, \mathsf{D}_{i+1})\\
&\geq 1 - 2\exp(-2Nm^2\varepsilon^2).
\end{aligned}$$

For the first factor we use that, conditional on $\mathsf{B}_{i+1/2}$ and $\mathcal{A}_i$, $\mathsf{p}_{i+1/2} \geq (1-\varepsilon)\mathsf{Iso}_N(\rho_i) \geq m(1-\varepsilon)$, and that each isolated particle lives independently from the others with probability $(1-\alpha_N)$; a similar estimate then gives

$$\mathbb{P}(\mathsf{C}_{i+1}|\mathsf{B}_{i+1/2}, \mathcal{A}_i, \mathsf{D}_{i+1}) \geq 1 - 2\exp(-2Nm^2(1-\alpha_N)^2\varepsilon^2(1-\varepsilon)^4).$$

The conclusion is that $\mathbb{P}(\mathcal{A}_{i+1}|\mathcal{A}_i) \geq (1-e^{-cN})^3$ for some c which depends on α_N and in ρ_0, and hence

$$\mathbb{P}(\tau_N \geq n+1) \geq (1-e^{-cN})^{3(n+1)} \tag{II.31}$$

as desired.

To obtain an estimate on the expected value appearing in the theorem we need to sum the right hand side of (II.31) in n. Note that in the above bounds, ε, δ and m are fixed, so we only need to understand how the constant c in (II.31) depends on α_N. Notice first that (since $\alpha \in [0,1)$) the dependence on α_N comes only from our bound on $\mathbb{P}(\mathsf{A}_0)$, namely $\mathbb{P}(\mathsf{A}_0) > 1 - \exp(-\alpha_N^2 N/2)$. If $\alpha_N \to \alpha \in (0,1)$ then there is nothing to proof. Otherwise, if $\alpha_N \to 0$, the condition $\alpha_N \log_2(N) \to \infty$ gives a similar bound, since fixing $M \in \mathbb{N}$ one gets $\alpha_N \geq \frac{M}{\log_2(N)}$ for large enough N, and then there exists $c' > 0$ such that $\mathbb{P}(\mathsf{A}_0) \geq 1 - \exp\left(-c'N(\log_2(N))^{-2}\right)$, which again gives us the bound we want. □

Proof of Theorem II.2.2.2(ii). We will just we explain how to adapt the proof of Theorem II.2.2.3 to obtain this result. We use the whole graph instead of a 3-independent set, and since the variables are now dependent, we change the Chernoff bounds to Chebychev bounds. Consequently we use $\mathsf{Isop}(x) = (1-e^{-\beta\rho})e^{-3\beta\rho}$ instead of Iso, and we bound the middle factor on the second line of (II.30) by

$$\begin{aligned}\mathbb{P}(\mathsf{B}_{i+1/2}|\mathcal{A}_i, \mathsf{D}_{i+1}) &= 1 - \mathbb{P}(\mathsf{B}^c_{i+1/2}|\mathcal{A}_i, \mathsf{D}_{i+1}) = 1 - \mathbb{E}(\mathbb{1}_{\mathsf{B}^c_{i+1/2}}|\mathcal{A}_i, \mathsf{D}_{i+1}) \\ &\geq 1 - \mathbb{E}\left(\frac{\mathsf{Var}(\mathsf{p}_{i+1/2})}{\varepsilon^2 \mathsf{Isop}(\rho_i)^2}|\mathcal{A}_i, \mathsf{D}_{i+1}\right) \geq 1 - \frac{10}{\varepsilon^2 N \underline{m}^2},\end{aligned}$$

where $\underline{m} := \min_{x\in[m(1-\alpha)(1-2\varepsilon),1-\alpha/2]} \mathsf{Isop}(x)$, and where the inequality is obtained from the site-percolation of parameter $f_\beta(\rho_i)$, at time i, and the following computation, which uses the independence between X^i_v and X^i_u for $u \in B(v,2)$.

$$\begin{aligned}\mathsf{Var}\left(\sum_{v\in G_N}\mathsf{X}^i_v\right) &= \mathbb{E}\left(\left(\sum_{v\in G_N}\mathsf{X}^i_v\right)^2\right) - \mathbb{E}\left(\sum_{v\in G_N}\mathsf{X}^i_v\right)^2 \\ &= \sum_{v\in G_N, u\in B(v,2)}\mathbb{E}\left(\mathsf{X}^i_v\mathsf{X}^i_u\right) + \sum_{v\in G_N, u\notin B(v,2)}\mathbb{E}\left(\mathsf{X}^i_v\mathsf{X}^i_u\right) - N^2\mathsf{Iso}(\rho_i) \\ &\leq (10N) + (N^2-10N)\mathsf{Iso}(\rho_i) - N^2\mathsf{Iso}(\rho_i) \\ &\leq 10N(1-\mathsf{Iso}(\rho_i)^2) \leq 10N\end{aligned}$$

(where the two summations in the second line correspond respectively to the first and second parenthesis in the third line). □

Now we turn to the results concerning the dynamical system $\mathsf{DS}(h)$. The following proposition will help us prove Proposition II.2.2.1.

Proposition II.4.0.2
DS(h) has 0 as a unique attractive fixed point if $\alpha \in [0,1]$, $\beta \in (0,\infty]$ satisfy

$$\phi(\alpha,\beta) \leq 1.$$

If this condition does not hold, i.e. $\phi(\alpha,\beta) > 1$, then 0 is a repulsive fixed point.

Proof. When the limit $\lim_{p\to 0} h'(p)$ is smaller than 1, the point 0 is attractive, while when this limit is bigger than 1, 0 is repulsive. A simple computation gives that

$$\lim_{p\to 0} h'(p) = \phi(\alpha,\beta)$$

We will show that if $\phi(\alpha,\beta) \leq 1$ then the identity function is always above h, yielding the uniqueness of the fixed point. This is enough to show that the orbits will converge to 0.

Let us compare the difference between the identity function and the function h. To begin we write $h(p) = g_\alpha(1-e^{-\beta p}) = (1-\alpha)(1-e^{-\beta p})G_\alpha(1-e^{-\beta p})^3$ so since the function G_α is decreasing in α (by Proposition II.5.1.2) it is enough to study the case $\alpha = 1-\frac{1}{\beta}$, meaning that our assertion is equivalent to proving that $p-(1-e^{-\beta p})G_{(1-1/\beta)}(1-e^{-\beta p})^3$ is positive for all $p \in (0,1]$ and for all $\beta > 1$. Again, since $G_{(1-1/\beta)}$ is decreasing in β, from Proposition II.5.1.2 in both the subindex $1-1/\beta$ and β, and since $\frac{1-e^{-\beta p}}{\beta}$ is also decreasing in β, it is enough to prove that the following holds:

$$p - \frac{\left(1-\sqrt{1-4e^{-p}(1-e^{-p})}\right)^3}{8(1-e^{-p})^2} > 0 \qquad \forall p \in (0,1].$$

But infimum of the right hand side of the last expression is obtained at 0 when $p \to 0$, and this yields the result. □

Proof of Proposition II.2.2.1. From Proposition II.5.1.2 for all $\alpha \in (0,1)$ the function g_α has a unique critical point x_0, which is a maximum. Without loss of generality we can suppose that $p \in [0, f_\beta^{-1}(p_\alpha^*)]$, otherwise apply h once to make this happen. Because it is the composition of two increasing functions, h is increasing inside $[0, f_\beta^{-1}(p_\alpha^*)]$. Using a restricted version of $h : [0, f_\beta^{-1}(p_\alpha^*)] \longrightarrow [0, f_\beta^{-1}(p_\alpha^*)]$, we get that 0 is the unique fixed point according to Proposition II.4.0.2. Finally, from [64, Prop. 2.3.5] the property holds in the first case. For the second case it is enough to notice that h is positive in $(0,1)$, hence the repulsive behavior of 0 in this regime yields the conclusion. □

II.5 Proofs of the multi-type results

As discussed in Section II.2, our approach to prove Theorem II.2.3.5 consists in using Theorem II.2.1.3 to show that te particle system “imitates" the behavior observed for the dynamical system in Theorems II.2.3.3 and II.2.3.1. However, in order to apply our approximation theorem, we need more information about DS(h) than just the definitions of *coexistence* and *domination*. These definitions only explicit the behavior of DS(h) in the long term, giving no control of the initial part of the orbits, where the randomness of the particle system might have a large impact. With this in mind we introduce a new property which will draw most of our attention in this section:

Definition II.5.0.1
We say that a set $A \subseteq [0,1]^2$ is *interior-trapping* for DS(h) if there are $0 < \delta' < \delta$ and $\bar{k} \in \mathbb{N}$ such that

(i) $\forall p \in A$, $d(p, A^c) > \delta \implies d(h(p), A^c) \geq \delta'$,

(ii) $\forall p \in A$, $d(p, A^c) \leq \delta \implies d(h^k(p), A^c) \geq \delta'$ for some $k \leq \bar{k}$.

In words, a set A is interior-trapping if the dynamical system cannot exit A using jumps larger than a certain size δ and if every time it gets to distance smaller than δ to the boundary, it takes a bounded number of steps for it to go back to the interior of A, where by *interior* here we mean a certain subset of A bounded away from the boundary (see Figure II.10).

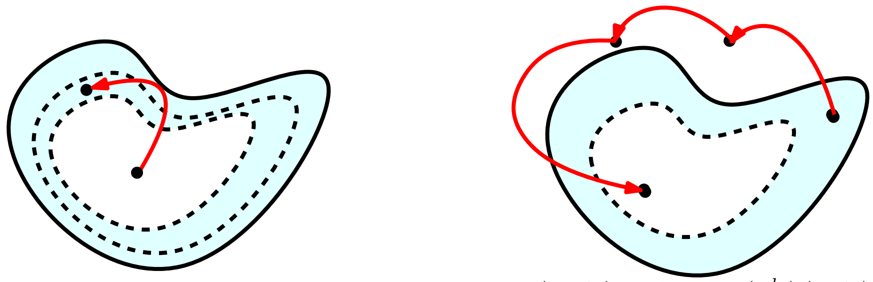


Figure II.10 – An interior-trapping set, where the white inner region is the set of points at distance larger than δ from the boundary

Interior-trapping sets as defined above are key within our techniques to show the connection between the particle system and DS(h). The next proposition shows that, since these sets provide some control over the amount of time the dynamical system spends near their boundary, we can use Theorem II.2.1.3 to provide a similar control for η_k^N.

Proposition II.5.0.2
Let $(\eta_k^N)_{k\in\mathbb{N}}$ be the mean-field MMM whose parameters satisfy the conditions in Theorem II.2.1.3, and assume that its initial condition ρ_0^N lies within an interior-trapping set A with parameters δ,δ' and $\bar{k}$. Then, there is $C > 0$ depending only on A such that

$$\mathbb{P}\left(\rho_k^N \notin A,\ \forall k \in \{1, 2, \ldots, \bar{k}\}\right) \leq C\theta_{\underline{\alpha}}(N). \tag{II.32}$$

Proof. Take the parameter $\delta' > 0$ from the definition of interior-trapping of A. From Theorem II.2.1.3, for this choice of δ' there is some $C > 0$ independent of ρ_0^N and N such that for any $k \leq \bar{k}$, we have

$$\mathbb{P}\Big(\delta' < \|\rho_k^N - h^k(\rho_0^N)\|\Big) \leq C\theta_{\underline{\alpha}}(N). \tag{II.33}$$

Observe now that from the interior-trapping property of A, we have only two possibilities for $d(\rho_0^N, A^c)$:

- If $d(\rho_0^N, A^c) \geq \delta$, then from Definition II.5.0.1.(i) we have $d(h(\rho_0^N), A^c) > \delta'$, so the left hand side of (II.32) is bounded by
$$\mathbb{P}(\rho_1^N \notin A) \leq \mathbb{P}(\delta' < \|\rho_1^N - h(\rho_0^N)\|) \leq C\theta_{\underline{\alpha}}(N).$$
- If $d(\rho_0^N, A^c) < \delta$, then from Definition II.5.0.1.(ii) there is $k \leq \bar{k}$ such that $d(h^k(\rho_0^N), A^c) > \delta'$,

so the left hand side of (II.32) is bounded by

$$\mathbb{P}(\rho_k^N \notin A) \;\le\; \mathbb{P}(\delta' < \|\rho_k^N - h^k(\rho_0^N)\|) \;\le\; C\theta_{\underline{\alpha}}(N).$$

On either case the bound is of the form $C\theta_{\underline{\alpha}}(N)$, giving the result. □

Using the newly defined property of interior-trapping we can state a result which shows that under the conditions of Theorems II.2.3.1 and II.2.3.3 we obtain a much stronger version of coexistence and domination.

Lemma II.5.0.3
Consider the dynamical system DS(h) with initial condition $p^0 \in (0,1)^2$:

1. (Coexistence) Assume that the parameters $\vec{\alpha}$ and $\vec{\beta}$ satisfy the conditions in Theorem II.2.3.1. Then, there is a compact interior-trapping set $A \subseteq (0,1)^2$ which contains p^0 as an interior point.
2. (Domination) Assume that the parameters $\vec{\alpha}$ and $\vec{\beta}$ satisfy the conditions in Theorem II.2.3.3. Then, there are $\gamma_1, \gamma_2 \in (0,1)$ and an interior-trapping set B (independent of p^0) with parameter $\bar{k} = 1$ such that $\forall p \in B$

$$(1-\alpha(1))f_{\vec{\beta}}^{(1)}(p) \le \gamma_1 p_1 \quad \text{and} \quad \gamma_2 < p_2. \tag{II.34}$$

Even further, there is $k \in \mathbb{N}$ such that $h^k(p^0)$ is an interior point of B.

Before turning to the proof of Lemma II.5.0.3 we show how this particular result gives all the results in Section II.2.3.

Proof of Theorem II.2.3.3. Under the assumptions of the theorem, Lemma II.5.0.3 states that the orbit of DS(h) eventually reaches an interior-trapping set B with parameter $\bar{k} = 1$, which satisfies (II.34) for some values $\gamma_1, \gamma_2 \in (0,1)$. Now, since $\bar{k} = 1$, from Definitions II.5.0.1.(i) and II.5.0.1.(ii) we deduce that the set B is actually trapping for the dynamical system, meaning that $h(p) \in B$ for all $p \in B$. Since B satisfies (II.34);

1. the condition $\gamma_2 < p_2$ for $p \in B$ implies that $\liminf_{k\to\infty} h_2^k(p) \ge \gamma_2 > 0$, and
2. the condition $h_1(p) \le (1-\alpha(1))f_{\vec{\beta}}^{(1)}(p_1) \le \gamma_1 p_1$ implies that $\lim_{k\to\infty} h_1^k(p) = 0$,

giving domination of the second type. □

Proof of Theorem II.2.3.1. Under the assumptions of the theorem, for any initial condition $p^0 \in (0,1)^2$ the lemma above gives a compact interior-trapping set $A \subseteq (0,1)^2$ containing p^0. From Definition II.5.0.1.(ii), every time the dynamical system leaves A, it spends at most $\bar{k}$ units of time in A^c, so the orbit of DS(h) is contained in $A_{\bar{k}} := \cup_{l=0}^{\bar{k}} h^l(A)$, which is compact from the compactness of A and the continuity of h. Since $A_{\bar{k}} \subseteq (0,1)^2$ (otherwise it would contain an orbit that never returns to A), compactness gives that it must be bounded away from the axes, so in particular for $i = 1,2$ we deduce that

$$\liminf_{k\to\infty} h_i^k(p) > 0,$$

giving coexistence. □

Proof of Theorem II.2.3.5. Consider the mean-field MMM and assume first that the parameters of the model satisfy the conditions of Theorem II.2.3.1. As in the proof of the previous theorem, Lemma II.5.0.3 gives a set $A \subseteq (0,1)^2$ containing ρ_0^N which is interior-trapping for the dynamical system DS(h). Defining $\sigma_1 = \inf\{k \geq 1, \rho_k^N \in A\}$ as the first return time of the dynamical system to A (with $\sigma_1 = \infty$ if there is no such k), it is clear that

$$\mathbb{P}(\sigma_1 > \bar{k}) \;=\; \mathbb{P}\left(\rho_k^N \notin A,\ \forall k \in \{1, 2, \ldots, \bar{k}\}\right),$$

which from Proposition II.5.0.2 is bounded by $C\theta_{\underline{\alpha}}(N)$ for some $C > 0$ which depends only on A. Since the bound is uniform on the initial condition, the strong Markov property gives that for any $n \in \mathbb{N}$,

$$\mathbb{P}\big(\sigma_1 \leq \bar{k},\ \sigma_2 - \sigma_1 \leq \bar{k}, \ldots,\ \sigma_n - \sigma_{n-1} \leq \bar{k}\big) \;\geq\; (1 - C\theta_{\underline{\alpha}}(N))^n$$

where each σ_n denotes the n-th return time to A. The event on the left hand side implies in particular that $\sigma_n < \infty$ a.s., but since $\sigma_n \geq n$ it follows that $\rho_k^N \in A$ for some $k \geq n$. Since both species have to be alive to lie within A, on this event both τ_N^1 and τ_N^2 must be larger than n, so we conclude (II.15).

To deduce (II.18) simply observe that from Lemma II.5.0.3 we can obtain an interior-trapping set A as before, but containing p as an interior point. If $\rho_0^N \to p$, then A contains all ρ_0^N for large N, so for all such N we can repeat the previous argument using the same interior-trapping set, which allows us to find some C' such that

$$\mathbb{P}(\tau_N^1, \tau_N^2 \geq n) \;\geq\; \left(1 - C'\theta_{\underline{\alpha}}(N)\right)^n$$

for all $N \in \mathbb{N}$. Taking $n = \theta_{\underline{\alpha}}(N)^{-(1-\varepsilon)}$ gives the result. (II.18).

We now turn our attention to the MMM whose parameters satisfy the conditions of Theorem II.2.3.3. Under this assumption Lemma II.5.0.3 gives an interior-trapping set B with parameter $\bar{k} = 1$ which satisfies (II.34). Choose $\gamma = \gamma_1$ where γ_1 is as in (II.34) and make the assumption that $\rho_0^N \in B$; we will show later how to treat the case $\rho_0^N \notin B$.

Since $\bar{k} = 1$, from Definition II.5.0.1 it is easy to see that regardless of the value of $d(\rho_0^N, B^c)$ we have $d(h(\rho_0^N), B^c) > \delta'$ so Theorem II.2.1.3 gives some $C > 0$ depending only on B such that

$$\mathbb{P}(\rho_1^N \notin B) \;\leq\; \mathbb{P}\big(\delta' < \big\|\rho_1^N - h(\rho_0^N)\big\|\big) \;\leq\; C\theta_{\underline{\alpha}}(N),$$

and since the bound is uniform over $\rho_0^N \in B$, an application of the strong Markov property gives that for any $n \in \mathbb{N}$,

$$\mathbb{P}(\rho_k^N \in B,\ \forall k \leq n) \;\geq\; (1 - C\theta_{\underline{\alpha}}(N))^n. \qquad \text{(II.35)}$$

Noticing that $\gamma_2 < p_2$ for all $p \in B$ we deduce that the event on the left hand side, which we call $\mathcal{E}_n$, implies $\tau_N^2 \geq n$ so we conclude (II.16).

To deduce (II.17) observe that by allowing the epidemic of the first species to attack but not to spread, the total amount of type 1 individuals at time 1 is a Poisson random variable with parameter $(1-\alpha_N(1))f_{\vec{\beta}}^{(1)}(\rho_0^N)$, so if $\rho_0^N \in B$, then this parameter is less than $\gamma_1\rho_0^N$. This way, it is easy to see that on the event $\mathcal{E}_n$, the process $(\rho_k^N)_{k\leq n}$ is stochastically dominated by a subcritical Galton-Watson process starting with $\rho_0^N N$ individuals and with offspring distribution Poisson$[\gamma_1]$. Using the classical results of Galton-Watson processes we easily deduce that

$$\mathbb{P}\left(\tau_N^1 \geq n\right) \leq \mathbb{P}\left(\mathcal{E}_n \cap \{\tau_N^1 > n\}\right) + \mathbb{P}(\mathcal{E}_n^c) \leq 1 - (1-\gamma^n)^N + 1 - (1 - C\theta_{\underline{\alpha}}(N))^n. \qquad \text{(II.36)}$$

Assume now that $\rho_0^N \notin B$ and observe that from Lemma II.5.0.3 there is some $k \in \mathbb{N}$ depending only on ρ_0^N such that $h^k(\rho_0^N)$ is an interior point of B. Let $\varepsilon > 0$ be small so that B contains the ball of center ρ_0^N with radius ε. Using Theorem II.2.1.3 there is some $\bar{C}$ depending on k and ε such that

$$\mathbb{P}(\rho_k^N \notin B) \le \mathbb{P}\Big(\varepsilon < \big\|\rho_k^N - h^k(\rho_0^N)\big\|\Big) \le \bar{C}\theta_{\underline{\alpha}}(N), \tag{II.37}$$

so the general proof of (II.16) and (II.17) follows from restricting to the event on the left hand side above and restarting the process at time k.

Finally, to conclude (II.19) observe that these inequalities would follow directly from (II.16), (II.17), and our definition of $\theta_{\underline{\alpha}}(N)$ if the parameter C was independent from ρ_0^N. However, neither the bound in (II.35) nor the one in (II.36) depend on ρ_0^N, so the only parameter dependent on ρ_0^N is $\bar{C}$ in (II.37). To show that under the additional assumption $\rho_0^N \to p$ we can find $\bar{C}$ independent from the initial condition, take $k \ge 0$ such that $h^k(p)$ is an interior point of B (such k exists because of Lemma II.5.0.3), and take ε such that B contains the ball centered at p with radius ε. Since $\rho_0^N \to p$, for large enough N we have $\|h^k(\rho_0^N) - h^k(p)\| < \frac{\varepsilon}{2}$, so from Theorem II.2.1.3 we obtain

$$\mathbb{P}(\rho_k^N \notin B) \le \mathbb{P}\Big(\varepsilon < \big\|\rho_k^N - h^k(p)\big\|\Big) \le \mathbb{P}\Big(\frac{\varepsilon}{2} < \big\|\rho_k^N - h^k(\rho_0^N)\big\|\Big) \le \bar{C}\theta_{\underline{\alpha}}(N), \qquad \square$$

for some $\bar{C}$ depending only on p.

The rest of this section is devoted to the proof of Lemma II.5.0.3 which is rather extensive and technical so we divide it into three subsections; in Section II.5.1 we present some notation and functions used to facilitate the analysis of the trajectories of DS(h), as well as some calculus results whose proofs are in Section II.5.3. Using these results we prove the coexistence statement of the lemma in Section II.5.2, and the extinction statement in Section II.5.3.

II.5.1 Preliminaries

We begin this section by decomposing the function g_α as

$$g_\alpha(x) = (1-\alpha)xG_\alpha(x)^3, \quad \text{with} \quad G_\alpha(x) = \frac{1-\sqrt{1-4(1-\alpha)x(1-x)}}{2(1-\alpha)x}.$$

This decomposition is useful since the function G_α satisfies the following properties whose proof we leave to the reader.

Proposition II.5.1.1
The function $G_\alpha : [0,1] \to [0,1]$ satisfies the following:

1. When $\alpha = 0$, it is defined by parts as
$$G_0(x) = \begin{cases} 1 & \text{if } x \le 1/2 \\ \frac{1-x}{x} & \text{if } x > 1/2 \end{cases}.$$
2. It is decreasing as a function of both α and x, with $G_\alpha(0) = 1$ and $G_\alpha(1) = 0$ for all $\alpha \in [0,1)$.
3. As $\alpha \to 1$, it converges monotonically to $G_1(x) := 1 - x$.

We now introduce $\bar{p} \in [0,1]^2$ as the maximum possible density achieved after the epidemic stage, that is

$$\bar{p}_i = \sup_{x\in[0,1]} g_{\alpha(i)}(x).$$

Since $g_\alpha \leq (1-\alpha)g_0$, it is easily seen that $\bar{p}_i \leq \frac{1-\alpha(i)}{2}$ and from its definition, except maybe for the initial value p^0, the orbit of DS(h) lies within $[0, \bar{p}]$, where we can use the next result to control the behavior of g_α:

Proposition II.5.1.2
There is a single value $x_0 \in [0, 1/2]$ where g_α attains its global maximum. This value is characterized as the solution of $G_\alpha(x_0) = x_0 + \frac{1}{2}$ and satisfies;

1. If $\alpha > 0$, this is the only critical point of g_α in $[0, 1]$.
2. If $\phi_i < 2\log 2$, then for any p with $p_i \leq \bar{p}_i$ we have $f^{(i)}_{\vec{\beta}}(p) < x_0$. In particular, $g'_{\alpha(i)} \circ f^{(i)}_{\vec{\beta}}(p) \geq 0$ for all $p \in [0, \bar{p}]$.

Even if g_α is not monotone, using the result above we can still obtain sufficient information about the growth of h:

Proposition II.5.1.3
For each $i = 1, 2$ define $l_i : [0, 1]^2 \to \mathbb{R}^+$ as $l_i(p) = h_i(p)/p_i$. Then:

1. The function $f^{(1)}_{\vec{\beta}}$ is increasing on p_1 and decreasing on p_2.
2. The function l_1 is decreasing on p_1.
3. If $\phi_1 < 2\log 2$, then h_1 is increasing on p_1 and decreasing on p_2. In particular, in this case l_1 is also decreasing on p_2.

The function l defined in Proposition II.5.1.3 is of great interest to us because from the relation $h_i(p) = l_i(p)p_i$ it is enough to bound l_i in order to show exponential growth or decay of a species. This is precisely what we do in the next result, which for simplicity we state under the assumption that the type 2 species is stronger than the type 1.

Proposition II.5.1.4
For any small $\varepsilon > 0$ define κ_ε as the unique solution of

$$g_{\alpha(1)}(1 - e^{-\beta(1)\kappa_\varepsilon}) \quad = \quad (1-\varepsilon)\kappa_\varepsilon.$$

Under the assumption $\phi_2 > \phi_1$ there are $\bar{c}, \varepsilon, \varepsilon' > 0$ small such that for all $c \leq \bar{c}$:

(i) For all $0 < p_1 < \kappa_\varepsilon$ it holds that:

$$p_2 \in (0, c) \Longrightarrow l_2(p) > 1 + \varepsilon'. \tag{II.38}$$
$$p_2 \in (c, \bar{p}_2) \Longrightarrow h_2(p) > (1 + \varepsilon')c. \tag{II.39}$$

(ii) Under the additional assumption $\phi_2 > 2\log 2$, the property above holds for all $p_1 > 0$.

(iii) If $\phi_1 < 2\log 2$, then:

$$p_1 \in (0, \kappa_\varepsilon) \Longrightarrow h_1(p) \leq (1 - \varepsilon')\kappa_\varepsilon. \tag{II.40}$$
$$p_1 \in (\kappa_\varepsilon, \bar{p}_1) \Longrightarrow l_1(p) \leq 1 - \varepsilon'. \tag{II.41}$$

The properties ((i)) and ((ii)) state that when the stronger species is at a very low density, it starts growing exponentially until it reaches a certain threshold value c, which becomes a lower bound for its density from that time onwards. Property ((iii)), on the other hand, states that if the fitness of the weaker species is below $2\log 2$, then its density decays exponentially until it reaches a trapping set $[0, \kappa_\varepsilon]$.

We are now ready to give the proof of coexistence in Lemma II.5.0.3.

II.5.2 Proof of Lemma II.5.0.3.1

From Proposition II.5.1.4 we already know that the stronger species survives, so in order to prove coexistence we need to prove the same for the weaker one. Assuming that the conditions of Theorem II.2.3.1 are satisfied, our approach consists in analyzing the dynamical system when the density of the weaker species is at low values. In that scenario we can approximate h by a simpler function $\bar{h}$, and show that for this particular dynamical system the density p_1 tends to grow on average.

We begin by defining the map $\bar{h} : [0,1]^2 \to [0,1]^2$ as

$$\bar{h}(p) = \begin{pmatrix} h_1(0,p_2) + p_1 \frac{\partial h_1}{\partial p_1}(0,p_2) \\ h_2(0,p_2) \end{pmatrix} = \begin{pmatrix} \phi_1 p_1 \frac{1-e^{-\beta(2)p_2}}{\beta(2)p_2} \\ h_2(0,p_2) \end{pmatrix}.$$

This linear approximation of h in the first component is intuitively good when taking p_2 fixed and then p_1 small. The next result, however, states the stronger assertion that the approximation is good uniformly on p_2:

Proposition II.5.2.1
For all $k \in \mathbb{N}$ we have

$$\lim_{p_1 \to 0} \frac{\bar{h}_1^k(p)}{h_1^k(p)} = 1$$

uniformly on $p_2 \in [0,1]$.

Proof. Define $\Sigma_k(p) := \beta(1)h_1^k(p) + \beta(2)h_2^k(p)$ and $\overline{\Sigma}_k(p) := \beta(2)\bar{h}_2^k(p)$. Using these values and the definition of h and $\bar{h}$ it is fairly simple to see that

$$\frac{\bar{h}_1^k(p)}{h_1^k(p)} = \frac{\bar{h}_1^{k-1}(p)}{h_1^{k-1}(p)} \cdot \left(\frac{\psi(\overline{\Sigma}_{k-1}(p))}{\psi(\Sigma_{k-1}(p))} \cdot (G_{\alpha(1)})^{-3} \circ f_{\vec{\beta}}^{(1)} \circ h_1^{k-1}(p) \right) \tag{II.42}$$

where ψ is defined as $\psi(x) = \frac{1-e^{-x}}{x}$, which is bounded away from zero if the argument x is bounded. Noticing that $\Sigma_k(p)$ and $\overline{\Sigma}_k(p)$ converge to the same value as $p_1 \to 0$, the expression between parentheses on the right hand side of (II.42) converges to 1 uniformly, so $\bar{h}_1^k(p)/h_1^k(p)$ converges to 1 uniformly if $\bar{h}_1^{k-1}(p)/h_1^{k-1}(p)$ does. Since $\bar{h}_1^0(p) = h_1^0(p) = p_1$, the result follows by repeating the argument k times. □

The proposition above allows us to approximate h by $\bar{h}$ whenever p_1 is small enough, where the definition of "small" does not depend on the value of p_2. The resulting dynamical system $(q^k)_{k\in\mathbb{N}}$

can be interpreted as running the one-dimensional MM for the type 2 species alone and then using its trajectory to compute the values of q_1^n as

$$q_1^n = q_1^0 \prod_{k=0}^{n-1} \frac{\phi_1(1-e^{-\beta(2)q_2^j})}{\beta(2)q_2^j}. \tag{II.43}$$

The expression above motivates us to study the observable $\bar{\varphi}$ of the orbit $(q_2^k)_{k\in\mathbb{N}}$, which is obtained as the limit when $n\to\infty$ of

$$\bar{\varphi}^n(x) = \frac{1}{n}\sum_{k=0}^{n-1}\varphi(h_2^k(0,x)),$$

where $\varphi\colon[0,1]\to[0,\infty)$ is defined as $\varphi(x)=\log\left(\frac{1-e^{-\beta(2)x}}{x}\right)$. The limit above exists because φ is decreasing, and since $\bar{p}_2\leq\frac{1}{2}$, we may also assume that it is bounded from below by $\log(2-2e^{-\beta(2)/2})$ which is positive from the assumption $\phi_2>z(\alpha(2))>2\log 2$. Notice that using $\bar{\varphi}^n$ we can write (II.43) as

$$q_1^n = q_1^0\left(\frac{\phi_1}{\beta(2)}e^{\bar{\varphi}^n(q_2^0)}\right)^n,$$

so the function $\frac{\phi_1}{\beta(2)}e^{\bar{\varphi}}$ represents the average growth of type 1 when taking into account the effect of type 2. To control this growth we define η to be the lowest possible value of $\bar{\varphi}$, that is

$$\eta = \inf_{x\in[0,1]}\bar{\varphi}(x).$$

The following result shows that bounding this term properly allows us to make q_1 grow to be as large as we want:

Lemma II.5.2.2

Suppose that the conditions of Theorem II.2.3.1 hold. If $\frac{\phi_1}{\beta(2)}e^{\eta}>1$, then for all $M>0$ there exists $\bar{k}\in\mathbb{N}$ satisfying the following property: For all $q_2^0\in[0,1]$, there is a $0\leq k\leq\bar{k}$ such that

$$\prod_{j=0}^{k-1}\frac{\phi_1(1-e^{-\beta(2)q_2^j})}{\beta(2)q_2^j} > M. \tag{II.44}$$

Proof. From the hypothesis we know that there exists $\delta>0$ such that $\phi_1=\beta(2)e^{-\eta}(1+2\delta)$. Taking $\varepsilon>0$ small enough such that $(1-\varepsilon)(1+2\delta)>1+\delta$, for each q_2^0 we can find $\underline{k}\in\mathbb{N}$ such that for all $k\geq\underline{k}$

$$\frac{\phi_1}{\beta(2)}\exp\left(\bar{\varphi}^k(q_2^0)\right) > (1-\varepsilon)\frac{\phi_1}{\beta(2)}\exp\left(\bar{\varphi}(q_2^0)\right) \geq (1-\varepsilon)\frac{\phi_1}{\beta(2)}e^{\eta} > 1+\delta,$$

where the first inequality follows from convergence of $\bar{\varphi}^k$ to $\bar{\varphi}$. Using the definition of $\bar{\varphi}^k$ we obtain

$$\frac{\phi_1}{\beta(2)}\left(\prod_{j=1}^{k-1}\frac{1-e^{-\beta(2)q_2^j}}{q_2^j}\right)^{1/k} > 1+\delta \qquad \forall k\geq\underline{k}. \tag{II.45}$$

In particular, since $1+\delta>1$, we find that for each q_2^0 there is some $k\geq\underline{k}$ such that

$$\left(\frac{\phi_1}{\beta(2)}\right)^k\prod_{j=0}^{k-1}\frac{1-e^{-\beta(2)q_2^j}}{q_2^j} > M.$$

For k fixed call O_k the set of all q_2^0 satisfying the inequality above for that given value of k. From the continuity of $\bar{h}$ each O_k is open, and from the previous argument, each q_2^0 belongs to some k, so $(O_k)_{k\in\mathbb{N}}$ is an open cover of $[0,1]$, so in particular it contains a finite subcover. Taking $\bar{k}$ to be the largest index of the subcover gives the result. □

The next result, in which we state the first notion of *interior-trapping* follows directly from Proposition II.5.2.1 and Lemma II.5.2.2.

Proposition II.5.2.3
Suppose that the conditions of Lemma II.5.2.2 are satisfied. If $p_2^0 \in (0,1)$, then there is $0 < \bar{c} < p_1^0$ small such that for all $c \leq \bar{c}$ we can find $\bar{k} \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$p_1^n \geq c \quad \Longrightarrow \quad \exists k \leq \bar{k} \text{ such that } p_1^{n+k} > \tfrac{3}{2r}c, \tag{II.46}$$

where r is defined as $r = \inf_{p\leq\bar{p}} l_1(p)$.

Another way to read this is that once the trajectory has one point above a certain threshold parameter c, then it will not stay below c more than $\bar{k}$ consecutive steps.

Proof of Proposition II.5.2.3:. Taking $M = \frac{2}{r^2}$ we obtain $\bar{k}$ as in Lemma II.5.2.2 and use the uniform convergence proved in Proposition II.5.2.1 to choose $\delta_0 > 0$ such that

$$p_1 < \delta_0 \quad \Longrightarrow \quad \frac{\bar{h}_1^k(p)}{h_1^k(p)} < \frac{4}{3} \quad \forall p_2 \in [0,1], \;\; \forall 1 \leq k \leq \bar{k}. \tag{II.47}$$

Having defined all these parameters, take $c = \min\{\frac{2}{3}\delta_0, \frac{1}{2}p_1^0\}$ so that in particular $c < p_1^0$. We prove (II.46) by contradiction as follows: Suppose that for some $n \in \mathbb{N}$ we have $p_1^n \geq c > p_1^{n+1}$ and that there is no $k \leq \bar{k}$ such that $p_1^{n+k} > \frac{3}{2r}c$. From our choice of c and (II.47), we have that each p_1^{n+k} is smaller than δ_0 so for each $k \leq \bar{k}$

$$p_1^{n+k} = h_1^k(p^n) \geq \frac{3}{4}\bar{h}_1^k(p^n) = \frac{3}{4}p_1^{n+1}\left(\frac{\phi_1}{\beta(2)}\right)^k \prod_{j=0}^{k-1} \frac{1-e^{-\beta(2)q_2^j}}{q_2^j}. \tag{II.48}$$

However, for the specific value of k given in Lemma II.5.2.2 with initial condition p_1^{n+1}, we can bound the right hand side in (II.48) by $\frac{3}{2r}p_1^{n+1}$. This is a contradiction with our assumption $p_1^{n+k} < \frac{3}{2r}c$ because

$$p_1^{n+k} > \frac{3}{2r^2}p_1^{n+1} = \frac{3}{2r^2}l_1(p_1^n)p_1^n \geq \frac{3}{2r^2}rc = \frac{3}{2r}c, \tag{II.49}$$

where the last inequality follows from our bound $l_1 \geq r$ and the assumption $p_1^n \geq c$. □

Using the results obtained so far, as well as Proposition II.5.1.4, we are finally ready to prove Lemma II.5.0.3.1. To do so observe that after one iteration, the orbit of DS(h) lies within $[0,\bar{p}]$ so we prove the lemma for the case where p^0 is in this set. First, we assume that the condition $\phi_1 e^{\eta} > \beta(2)$ is satisfied and for $r = l_1(\bar{p})$ take c_1 as in Proposition II.5.2.3. Next, observe that from the hypotheses of Theorem II.2.3.1 we have that $\phi_2 > z(\alpha(2)) > 2\log 2$ so we can take c_2 as in Proposition II.5.1.4.(ii). We claim that the set

$$A := [c_1, \bar{p}_1] \times [c_2, \bar{p}_2]$$

satisfies the properties needed in the lemma. Indeed, it is clear that the set is compact and contains p^0 as an interior point. To see that the set is interior-trapping, notice that from (II.39) in Proposition II.5.1.4, for any $p_2 \in (c_2, \bar{p}_2)$ we have $h_2(p_2) > (1+\varepsilon')c_2$ independently from p_1, so that both requirements for interior-trapping are satisfied with $\bar{k}=1$ in the second component. To deduce the same for the first component notice that from the definition of r, we have

$$p_1 > \frac{c}{r} \quad \Longrightarrow \quad p_1^1 > c$$

and from Proposition II.5.2.3 there is $\bar{k}$ such that

$$\frac{c}{r} > p_1 \geq c \quad \Longrightarrow \quad \exists k \leq \bar{k} \text{ such that } p_1^k > \tfrac{3}{2r}c$$

so both requirements for the interior-trapping definition are satisfied in this component as well.

Finally, to conclude the result we need to show that the hypotheses assumed in Theorem II.2.3.1 imply $\phi_1 e^{\eta} > \beta(2)$. To see this notice that the φ is a decreasing function so we can bound it by taking x as large as possible. On the other hand, for any value of p_2^0 we have that $p_2^k \leq \bar{p}_2 \leq \frac{1-\alpha(2)}{2}$, so in particular

$$\varphi(p_2^k) \geq \varphi\left(\tfrac{1-\alpha(2)}{2}\right) = \log\left(\tfrac{2(1-e^{-\phi_2/2})}{1-\alpha(2)}\right), \tag{II.50}$$

so the term on the right is a lower bound for η. We can improve this bound reasoning as follows: Take κ_2 as defined in Theorem II.2.3.1 so that in particular $P_2 := \frac{\kappa_2}{\beta(2)}$ is a fixed point of $h_2(0,\cdot)$. Now, since the function $x \to 1-e^{-\beta(2)x}$ is increasing, from Proposition II.5.1.2 it follows that the function $h_2(0,\cdot)$ has a unique critical point P_1, in which it attains its maximum. Assume for the moment that $P_1 \leq P_2$; under this assumption it follows that $h_2(0,\cdot)$ is decreasing on $[P_2, \frac{1}{2}]$ so that

$$P_2 \leq x \quad \Longrightarrow \quad h_2(0,x) \leq h_2(0,P_2) = P_2.$$

In words, every point q_2^k in the orbit of q_2 which is larger than P_2 is followed by an element q_2^{k+1} smaller than P_2 so at least half of the points in the orbit lie in $[0,P_2]$. Bounding by $\varphi(P_2)$ the value of φ in this interval, and by $\varphi(\frac{1-\alpha(2)}{2})$ the value outside of it, we obtain

$$\bar{\varphi}(p_2^0) \;\geq\; \tfrac{1}{2}\left[\log\left(\tfrac{1-e^{-\beta(2)P_2}}{P_2}\right) + \log\left(\frac{2-2e^{-\phi_2/2}}{1-\alpha(2)}\right)\right] \qquad \forall p_2^0 \in [0,1/2].$$

Since this bound is satisfied for all p_2^0, it is also a lower bound for η, but it is easily seen that with this bound, $\phi_1 > e^{-\eta}\beta(2)$ is equivalent to condition (II.13).

Finally, it only remains to show that $P_1 \leq P_2$; but since P_1 is the argument of the only maximum of $h_2(0,\cdot)$ and P_2 is a fixed point of this function, it is not hard to be convinced that the inequality is equivalent to

$$P_1 \leq g_{\alpha(2)}\left(1-e^{-\beta(2)P_1}\right). \tag{II.51}$$

From Proposition II.5.1.2, $x_0 = 1-e^{-\beta(2)P_1}$ is a critical point of $g_{\alpha(2)}$ so it satisfies $G_{\alpha(2)}(x_0) = x_0 + \frac{1}{2}$. Replacing these equalities into (II.51) we obtain that x_0 must satisfy

$$\phi_2 x_0 (x_0 + \tfrac{1}{2})^3 + \log(1-x_0) \geq 0.$$

Now, from Proposition II.5.1.1, the equality $G_{\alpha(2)}(x_0) = x_0 + \frac{1}{2}$ defines an implicit function $x_0(\alpha(2))$ which is strictly decreasing in $\alpha(2)$ and satisfies $x_0(0) = 1/2$ and $x_0(1) = 1/4$. Solving for ϕ_2, the inequality above becomes

$$\phi_2 > z(\alpha(2)) := \frac{-\log(1-x_0)}{x_0(x_0+1/2)^3}, \tag{II.52}$$

which is satisfied by our hypothesis on ϕ_2 so we conclude that $P_1 \leq P_2$ and the result then follows.

II.5.3 Proof of Lemma II.5.0.3.2

We want to prove that there is a trapping set B where the stronger species survives while the density of the weaker one decays exponentially. The corner stone of this section is the following lemma, which shows that the set where p_2 remains bounded away from zero and p_1 decreases, is a trapping set.

Lemma II.5.3.1
Assume that conditions of Theorem II.2.3.3 hold. Take $\bar{c}$, ε and ε' as in Proposition II.5.1.4 and for sufficiently small $c < \bar{c}$ let

$$B_1 \;=\; \Big\{p \in [0, \kappa_\varepsilon] \times [c, \bar{p}_2],\ l_1(p) < 1\Big\},$$

where κ_ε is defined in Proposition II.5.1.4 as the solution of $g_{\alpha(1)}(1 - e^{-\beta(1)\kappa_\varepsilon}) = (1 - \varepsilon)\kappa_\varepsilon$. Then

$$\sup_{p \in B_1} l_1 \circ h(p) < 1 \quad \text{and} \quad \inf_{l_1(p) \geq 1} l_2(p) > 1. \tag{II.53}$$

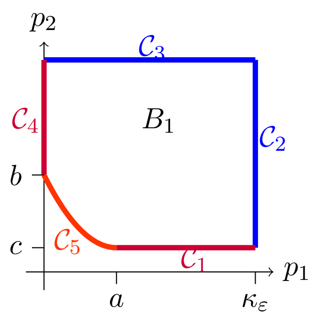
Proof. We begin by observing that $\phi_1 < 2\log 2$. Indeed, from the assumption $\phi_1 < \phi_2$ the result follows if $\phi_2 \leq 2\log 2$. Assume then that $\phi_2 > 2\log 2$ and observe that condition (II.14) gives

$$a_1(\phi_1) \;<\; \frac{\phi_2}{1 - \alpha(2)} g_{\alpha(2)}(1 - e^{-\frac{\phi_2}{2}}) \;\leq\; 8\phi_2(1 - e^{-\frac{\phi_2}{2}})e^{-\frac{3\phi_2}{2}},$$

where we have used that $G_{\alpha(2)}(x) \leq 2(1-x)$. Now, the function on the right hand side is decreasing in $(2\log 2, +\infty)$, so we can bound $a_1(\phi_1)$ by $16\log 2(1-e^{-\frac{2\log 2}{2}})e^{-3\log 2} = \log 2$, and thus $\phi_1 < 2\log 2$, using the definition and monotonicity of $a_1(x)$. This bound on ϕ_1 will allow us to use conclude from Proposition II.5.1.3 the monotonicity of h_1.

From Proposition II.5.1.3 we also know that l_1 is strictly decreasing on both p_1 and p_2, so the level set $\{l_1(p) = 1\}$ defines a strictly decreasing function $p_2 = s(p_1)$, for which there are values a and b such that $l_1(a, c) = l_1(0, b) = 1$. Using these values we can easily characterize B_1 as a set bounded by the curves

$$\begin{aligned}
\mathcal{C}_1 &:= \{(p_1, c),\ a \leq p_1 \leq \kappa_\varepsilon\}\\
\mathcal{C}_2 &:= \{(\kappa_\varepsilon, p_2),\ c \leq p_2 \leq \bar{p}_2\}\\
\mathcal{C}_3 &:= \{(p_1, \bar{p}_2),\ 0 \leq p_1 \leq \kappa_\varepsilon\}\\
\mathcal{C}_4 &:= \{(0, p_2),\ b \leq p_2 \leq \bar{p}_2\}\\
\mathcal{C}_5 &:= \{(p_1, s(p_1)),\ 0 \leq p_1 \leq a\}
\end{aligned}$$



(the curve $\mathcal{C}_5$ is represented in the figure by an arbitrary decreasing function). We will make use of the following lemma, its proof is postponed.

Lemma II.5.3.2

$$\sup_{p \in B_1} l_1 \circ h(p) \;=\; \max_{p \in \mathcal{C}_1 \cup \mathcal{C}_4 \cup \mathcal{C}_5} l_1 \circ h(p). \tag{II.54}$$

Thus in order to obtain the first statement in (II.53) we need to find the maximum of $l_1 \circ h$ on each set $\mathcal{C}_1$, $\mathcal{C}_4$ and $\mathcal{C}_5$ separately.

• $\mathcal{C}_1$: From Proposition II.5.1.3 we know that $l_1(\cdot,0)$ is strictly decreasing and since $\phi_1 < 2\log 2$, the same proposition states that $h_1(\cdot,0)$ is strictly increasing on $(0,\kappa_\varepsilon]$. As a result, the function $p_1 \mapsto l_1(h_1(p_1,0),0)$ is strictly decreasing with no critical points on any interval $[u,\kappa_\varepsilon]$, so its derivative is negative and bounded away from zero. Since all the functions are smooth, if c is sufficiently small, we also obtain that $\frac{\partial}{\partial p_1} l_1 \circ h$ is negative and bounded away from zero on $\mathcal{C}_1$. We conclude that $l_1 \circ h$ is maximized at the point (a,c), so we need to show that its value at that point is less than one. Indeed, using the definition of a, we obtain $h_1(a,c) = a$. Also, since $a < \kappa_\varepsilon$ we can use Proposition II.5.1.4 to deduce that $h_2(a,c) > c$, where the inequality follows from Proposition II.5.1.4. From the monotony of l_1 we finally deduce $l_1 \circ h(a,c) < l_1(a,c) = 1$.

• $\mathcal{C}_4$: In this set we have $p_1 = 0$, which greatly simplifies the analysis since

$$h_1(0,p_2) = 0, \qquad h_2(0,p_2) = g_{\alpha(2)}(1 - e^{-\beta(2)p_2}), \qquad l_1 \circ h = \phi_1 \tfrac{1-e^{-\beta(2)h_2}}{\beta(2)h_2}.$$

Indeed, from the particular form of $l_1 \circ h$ on this set, the condition $l_1 \circ h < 1$ is equivalent to $\frac{1-e^{-\beta(2)h_2}}{\beta(2)h_2} < \frac{1-e^{-a_1(\phi_1)}}{a_1(\phi_1)}$ from the definition of $a_1(\phi_1)$. Now, since the function $\frac{1-e^{-x}}{x}$ is decreasing we obtain

$$l_1 \circ h(0,p_2) < 1 \quad \iff \quad a_1(\phi_1) < \beta(2) g_{\alpha(2)}(1 - e^{-\beta(2)p_2}). \tag{II.55}$$

Observe now that l_1 is a decreasing function, so it is maximized at the points where h_2 attains its minimum. From the special form of h_2 given above, we deduce from Proposition II.5.1.2 that h_2 is minimized either where p_2 is maximal or minimal. Following this argument, we conclude that the maximum value of l_1 on $\mathcal{C}_4$ is either $l_1 \circ h(0,\bar{p}_2)$ or $l_1 \circ h(0,b)$.

When applying the equivalence (II.55) to the term $l_1 \circ h(0,\bar{p}_2)$ we obtain that it is smaller than one if and only if

$$a_1(\phi_1) < \beta(2) g_{\alpha(2)}(1 - e^{-\beta(2)\bar{p}_2}),$$

which follows from $\bar{p}_2 < \frac{1-\alpha(2)}{2}$ and (II.14). On the other hand, from $l_1(0,b)$ it is easy to see that $a_1(\phi_1) = \beta(2)b$ so the term $l_1 \circ h(0,b)$ is smaller than one iff

$$a_1(\phi_1) < \beta(2) g_{\alpha(2)}(1 - e^{-a_1(\phi_1)}),$$

but this follows directly from (II.14).

• $\mathcal{C}_5$: On this set it will be enough to show that

$$\inf_{p\in\mathcal{C}_5} \left[\phi_2 G^3_{\alpha(2)} \circ f^{(2)}_{\vec{\beta}} - \phi_1 G^3_{\alpha(1)} \circ f^{(1)}_{\vec{\beta}}\right](p) > 0. \tag{II.56}$$

Indeed, if (II.56) is satisfied then multiplying the inequality by $\frac{1-e^{-\Sigma_p}}{\Sigma_p}$ gives $l_2(p) > l_1(p) = 1$, where $\Sigma_p = \beta(1)p_1 + \beta(2)p_2$. Now,

$$l_2(p) > 1 \implies p_2 < h_2(p) \implies l_1(h) = l_1(p_1,h_2) < l_1(p) = 1,$$

so (II.56) implies $l_1 < 1$. To prove (II.56) recall that $s(p_1)$ is a decreasing function, which means that $f^{(1)}_{\vec{\beta}}(p_1,s(p_1))$ is increasing and $f^{(2)}_{\vec{\beta}}(p_1,s(p_1))$ is decreasing. It follows that on $\mathcal{C}_5$ the function in (II.56) is increasing on p_1, so the infimum is positive if the inequality holds at $(0,b)$ which in this case follows from assumption (II.14).

To complete the proof we need to show that $\inf_{l_1(p)\geq 1} l_2(p) > 1$, but l_2 is decreasing on p_2 and the maximum values of p_2 within the region given by $l_1 \leq 1$ are found whenever $l_1 = 1$. This way, it is enough to show that $\inf_{l_1(p)=1} l_2(p) > 1$, but this is analogous to the proof of (II.56), so the result follows. □

To complete the result above we need to prove Lemma II.5.3.2, which follows from a monotonicity argument similar to the ones used before.

Proof of Lemma II.5.3.2. Observe that, since $f^{(2)}_{\vec{\beta}}$ is increasing in p_2 and decreasing in p_1, the level sets $\{f^{(2)}_{\vec{\beta}}(p) = \gamma\}$ define strictly increasing functions $p_2 = r_\gamma(p_1)$. On these level sets h_2 is clearly constant and h_1 is increasing in p_1; this last statement follows from the monotonicity of $g_{\alpha(1)}$ (proved in Proposition II.5.1.2) and from $f^{(1)}_{\vec{\beta}}(p_1, r_\gamma(p_1)) + \gamma = (f^{(1)}_{\vec{\beta}} + f^{(2)}_{\vec{\beta}})(p_1, r_\gamma(p_1)) = 1 - \exp(-\beta(1)p_1 - \beta(2)r_{\gamma(p_1)})$, which implies that $f^{(1)}_{\vec{\beta}}$ increases in p_1. Since l_1 is decreasing in both arguments, at each level set, $l_1(h)$ attains its maximum at points of minimal values of p_1. Our claim then is a result of the fact that each point $p \in A$ belongs to a level set $f^{(2)}_{\vec{\beta}} \equiv \gamma$ which attains a minimal value of p_1 at $\mathcal{C}_1 \cup \mathcal{C}_4 \cup \mathcal{C}_5$. □

The rest of the proof of Lemma II.5.0.3.2 consists of modifying B_1 until obtaining the interior-trapping set B required in the lemma. As a first step, observe that from Lemma II.5.3.1 there is some $\gamma \in (0,1)$ such that $\sup_{p\in B_1} l_1 \circ h(p) = \gamma$. We will build an interior-trapping set B_2 simply by modifying a little bit the definition of B_1. Define

$$B_2 := \left\{p \in [0, \kappa_\varepsilon] \times [c, \bar{p}_2],\ l_1(p) < \bar{\gamma}\right\}$$

for some $\bar{\gamma} \in (\gamma, 1)$. We claim that this set is interior-trapping with parameter $\bar{k} = 1$. Indeed, take any $p \in B_2$, then, from our choice of parameters:

- From Proposition II.5.1.4.(iii) we have $h_1(p) \leq (1-\varepsilon')\kappa_\varepsilon$.
- Since $p_1 \leq \kappa_\varepsilon$, from Proposition II.5.1.4.(i) we have $h_2(p) \leq (1+\varepsilon')c$.
- From Lemma II.5.3.1 we have $\sup_{p\in B_2} l_1 \circ h(p) \leq \sup_{p\in B_1} l_1 \circ h(p) = \gamma$.

This way, there is some $\delta > 0$ such that $d(h(p), B_2^c) > \delta$ uniformly on $p \in B_2$, which proves that the set is interior-trapping with parameter $\bar{k} = 1$. To show that the dynamical system reaches B_2, it suffices to show that it reaches B_1 in finite time. Fix an initial condition p^0. If $p_1^0 > \kappa_\varepsilon$, then by Proposition II.5.1.4.(iii) we have $p_1^1 \leq (1-\varepsilon')p_1^0$ and we repeat the argument until the trajectory reaches $[0, \kappa_\varepsilon] \times [0, \bar{p}_2]$, where it remains forever. From this point on we assume that $p_2^0 > c$, since if this is not satisfied we use Proposition II.5.1.4.(i) to obtain $p_2^1 > p_2^0(1+\varepsilon')$, and then repeat the argument to show that the sequence eventually reaches $[0, \kappa_\varepsilon] \times [c, \bar{p}_2]$, where it remains forever. Hence to finish the proof it is enough to consider an initial condition p^0 inside this set and show that there is some finite k such that $l_1(p^k) < 1$. Suppose this is not the case. Then for all $n \in \mathbb{N}$ we have $l_1(p^n) \geq 1$, but from Lemma II.5.3.1 this implies that there is some $\varepsilon > 0$ such that $l_2(p_n) > 1 + \varepsilon$ for all n. In particular, $p_2^{n+1} > (1+\varepsilon)p_2^n$ for all n and hence $p_2^n \to \infty$, which is impossible since $p_2 \in [0, 1]$. We conclude that the dynamical system reaches B_1.

It remains to show that there are γ_1 and γ_2 such that

$$(1-\alpha(1))f^{(1)}_{\vec{\beta}}(p) \leq \gamma_1 p_1 \quad \text{and} \quad \gamma_2 < p_2.$$

Taking $\gamma_2 = c$ the inequality on the right is trivially satisfied. The main problem is that in B_2 the decay inequality is of the form $h_1(p) \leq \bar{\gamma} p_1$, which is not as strong as the one needed. However, once inside B_2 we have $p_1^k \to 0$, so in particular it is easy to see that for each δ, the set $B_\delta \subseteq B_2$ given by

$$B_\delta := \left\{ p \in [0, \delta] \times [c, \bar{p}_2],\ l_1(p) < \bar{\gamma} \right\}$$

is also interior-trapping and satisfies the same properties as B_2. Indeed, once the dynamical system reaches B_2, p_1^n decreases exponentially so it reaches B_δ. For any $\varepsilon > 0$ we can take δ sufficiently small, so that for any $p_1 < \delta$ we have

$$G^3_{\alpha(1)} \circ f^{(1)}_{\vec{\beta}}(p) \geq 1 - \varepsilon.$$

Choosing ε sufficiently small, we use the inequality above to conclude that

$$(1 - \alpha(1)) f^{(1)}_{\vec{\beta}}(p) \leq \frac{\bar{\gamma}}{1 - \varepsilon} p_1$$

and the result then follows taking $\gamma_1 = \frac{\bar{\gamma}}{1-\varepsilon}$.

Appendix

Contents

II.A Proof of Lemma II.3.0.2

For each $k \geq 2$ define

$$W_k = \sqrt{\mathbb{E}_1\big((1-\alpha_N)^{Z_2+\cdots+Z_k}\big)}$$

where $\mathbb{E}_1$ stands for the law of the Galton-Watson process with $Z_1 = 1$. Since Z_0 is a Bernoulli random variable with parameter q, we clearly have (with the obvious notation)

$$\begin{aligned}
\mathbb{E}\big(Z_0(1-\alpha_N)^{Z_0+\cdots+Z_{L_N-1}}\big) &= q(1-\alpha_N)\mathbb{E}\big((1-\alpha_N)^{Z_1+\cdots+Z_{L_N-1}}\big) \\
&= q(1-\alpha_N)\mathbb{E}\Big((1-\alpha_N)^{Z_1}\big(\mathbb{E}_1(1-\alpha_N)^{Z_2+\cdots+Z_{L_N-1}}\big)^{Z_1}\Big) \\
&= q(1-\alpha_N)r((1-\alpha_N)W_{L_N-1}^2)
\end{aligned}$$

where $r(x) = (qx+1-q)^3$ is the probability generating function of a Binomial$[3, q]$ random variable. To obtain an expression for W_{L_N-1} we study the sequence $(W_k)_{k\geq 2}$ which, using the same reasoning as above, satisfies the quadratic recurrence equation

$$W_{k+1} = q(1-\alpha_N)W_k^2 + 1 - q \tag{II.57}$$

with initial condition $W_2 = (1-\alpha_N)q + 1 - q$. This recurrence equation has two fixed points,

$\frac{1\pm\sqrt{1-4q(1-q)(1-\alpha_N)}}{2q(1-\alpha_N)}$, being the one with a positive sign repulsive, and the one with a

minus sign attractive, so all orbits starting in $[0, 1]$ converge to the latter which we call $\overline{W}$. From its definition we have

$$r((1-\alpha_N)\overline{W}^2) = \Big[q(1-\alpha_N)\overline{W}^2 + 1 - q\Big]^3 = \overline{W}^3,$$

and observing that $g_{\alpha_N}(q) = q(1-\alpha_N)\overline{W}^3$, we deduce that (II.27) is equivalent to

$$q(1-\alpha_N)\Big|r((1-\alpha_N)W_{L_N-1}^2) - r((1-\alpha_N)\overline{W}^2)\Big| \leq C\theta_\alpha(N). \tag{II.58}$$

Since $q(1-\alpha_N) \leq 1$ and $|r(a)-r(b)| \leq 3|a-b|$ for all $a, b \in [0,1]$, it will be enough to show that $|W_{L_N-1} - \overline{W}| \leq C\theta_\alpha(N)$. To this end we notice that, from the definition of $\overline{W}$,

$$\begin{aligned} |W_{k+1} - \overline{W}| &= \left| [q(1-\alpha_N)W_k^2 + 1 - q] - [q(1-\alpha_N)\overline{W}^2 + 1 - q] \right| \\ &= q(1-\alpha_N)|W_k - \overline{W}|(W_k + \overline{W}) \\ &\leq q(1-\alpha_N)|W_k - \overline{W}|(1 + \overline{W}), \end{aligned} \tag{II.59}$$

but it can be easily deduced that $q(1+\overline{W}) \leq 1$, thus

$$|W_{k+1} - \overline{W}| \leq (1-\alpha_N)|W_k - \overline{W}| \tag{II.60}$$

for all $k \geq 2$. In particular, we obtain

$$|W_{L_N-1} - \overline{W}| \leq 2(1-\alpha_N)^{L_N-2} \leq Ce^{-\alpha_N L_N} = CN^{-\frac{\alpha_N}{5\log(2)}},$$

where the last equality follows from the definition of L_N. On the one hand, if $\alpha \neq 0$, then for N large the exponent is smaller than $-\frac{\alpha}{5}$ giving the result. On the other hand, when $\alpha = 0$, we need to improve this bound. To do so, we use (II.60) to bound the distance between the $\frac{L_N}{2}$-th term of the sequence and $\overline{W}$, obtaining the similar expression;

$$|W_{L_N/2} - \overline{W}| \leq 2e^{-\frac{\alpha_N(L_N-2)}{2}} \leq Ce^{2\log\alpha_N} = C(\alpha_N)^2,$$

where in the second inequality we used condition (II.8) to bound the exponent (this is valid for N large, hence the C factor). Noticing that W_k converges monotonically to $\overline{W}$, the bound above is valid for all W_k with $k \geq \frac{L_N}{2}$, thus we can restart the sequence at the $\frac{L_N}{2}$-th term to improve the bound taken in (II.59) as

$$\begin{aligned} |W_{k+1} - \overline{W}| &= q(1-\alpha_N)|W_k - \overline{W}|(W_k + \overline{W}) \\ &\leq q(1-\alpha_N)|W_k - \overline{W}|(C(\alpha_N)^2 + 2\overline{W}). \end{aligned}$$

But $2q(1-\alpha_N)\overline{W} = 1 - \sqrt{1-4q(1-q)(1-\alpha_N)} \leq 1 - \sqrt{\alpha_N}$, giving

$$|W_{k+1} - \overline{W}| \leq |W_k - \overline{W}|[1 - \sqrt{\alpha_N} + C(\alpha_N)^2]$$

for all $k \geq \frac{L_N}{2}$. In particular,

$$|W_{L_N-1} - \overline{W}| \leq 2[1 - \sqrt{\alpha_N} + C(\alpha_N)^2]^{L_N/2} \leq Ce^{-\frac{\sqrt{\alpha_N}\log N}{20}} \leq Ce^{-\sqrt{\log N}},$$

where we used that $\alpha_N \log N \to \infty$ as $N \to \infty$.

II.B Proof of Proposition II.5.1.2

We prove only the case $\alpha > 0$ since the case $\alpha = 0$ is much easier to handle. Observe first that $G_\alpha(x)$ satisfies

$$G_\alpha(x)\sqrt{1-4(1-\alpha)x(1-x)} = -G_\alpha(x) + 2 - 2x. \tag{II.61}$$

$$G'_\alpha(x) = \frac{G_\alpha(x)-1}{x\sqrt{1-4(1-\alpha)x(1-x)}} = \frac{G_\alpha(x)-1}{x[1-2(1-\alpha)xG_\alpha(x)]}. \tag{II.62}$$

To obtain the maximum of g_α we impose the first order condition

$$0 = g'_\alpha(x) = xG^3_\alpha(x)\left[\frac{1}{x} + \frac{3G'_\alpha(x)}{G_\alpha(x)}\right],$$

where the term $xG^3_\alpha(x)$ is equal to 0, only at 0 and 1, so $g'_\alpha(x) = 0$ only if the term on the right vanishes. It is left to the reader that together with (II.61) and (II.62), the condition above gives $G_\alpha(x) = x + 1/2$. This way, since $G_\alpha \le 1$, every critical point of the function must lie in $[0, 1/2]$.

The first part of the proposition will follow if we show that at every critical point x_0 we have $g''_\alpha(x_0) < 0$, that is, every critical point is a maximum (hence there can be only one). Now, since $g'_\alpha(x_0) = 0$,

$$g''_\alpha(x_0) \quad = \quad g_\alpha(x_0)\left[\frac{3G''_\alpha(x_0)}{G_\alpha(x_0)} - \frac{4}{3x_0^2}\right],$$

and it is enough to show that $G''_\alpha(x_0) < 0$. Deriving G_α twice and using (II.61) and (II.62) we obtain the expression

$$G''_\alpha(x) \quad = \quad \frac{[G_\alpha(x) - 1]2(1-\alpha)x[2G_\alpha(x) + xG'_\alpha(x)]}{[x(1 - 2(1-\alpha)xG_\alpha)]^2},$$

which is negative as soon as $2G_\alpha(x_0) + xG'_\alpha(x_0) > 0$ since $G_\alpha \le 1$. To check the inequality we use (II.61) and (II.62) yet again to show that it is equivalent to $3 - 4x > G_\alpha(x_0)$, which is satisfied because $0 \le x \le 1/2$.

To prove the second part of the proposition, let us denote $\Sigma_p := \beta(1)p_1 + \beta(2)p_2$, so that for each $i = 1, 2$ we can write

$$f^{(i)}_{\vec{\beta}}(p) \quad = \quad \frac{1 - e^{-\Sigma_p}}{\Sigma_p}\beta(i)p_i.$$

Since the function $x \mapsto \frac{1-e^{-x}}{x}$ is decreasing, it follows that $f^{(i)}_{\vec{\beta}}(p) \le 1 - e^{-\beta(i)p_i} \le 1 - e^{-\beta(i)g_{\alpha(i)}(x_0)}$ so it will be enough to prove that $1 - e^{-\beta(i)g_{\alpha(i)}(x_0)} \le x_0$ or, equivalently $\beta(i)g_{\alpha(i)}(x_0) \le -\log(1 - x_0)$. Since x_0 is characterized by $G_{\alpha(i)}(x_0) = x_0 + 1/2$, it will be enough to show the following inequality:

$$V(x_0) \;:=\; \phi_i x_0\left(\tfrac{1}{2} + x_0\right)^3 + \log(1 - x_0) \;\le\; 0. \tag{II.63}$$

Our approach to show (II.63) is to prove that the function V is non-positive on the entire interval $[0, 1/2]$. Indeed, $V(0) = 0$ and $V(1/2) = \frac{\phi_i}{2} - \log 2$, which is negative from our assumption $\phi_i < 2\log 2$, so it is enough to prove that the inequality holds at the critical points of V; this follows from

$$V'(x) = \phi_i(\tfrac{1}{2} + x)^2(\tfrac{1}{2} + 4x) - \tfrac{1}{1-x}, \qquad V''(x) = \phi_i(\tfrac{1}{2} + x)(3 + 12x) - \tfrac{1}{(1-x)^2},$$

so whenever $V'(x_1) = 0$ we have $(1 - x_1)V''(x_1) = \phi_i(x_1 + 1/2)[-16x_1^2 + 13x_1/2 + 11/4]$, which is positive in $[0, 1/2]$, giving that x_1 is a minimum.

II.C Proof of Proposition II.5.1.3

Maintaining the notation Σ_p used in the previous proof;

1. For the dependence of $f^{(1)}_{\vec{\beta}}$ on p_1 we write the function as $(1-e^{-\Sigma_p})\frac{\beta(1)p_1}{\Sigma_p}$, which, for fixed p_2 is the product of two increasing functions. For the dependence of $f^{(1)}_{\vec{\beta}}$ on p_2 on the other hand, we write $f^{(1)}_{\vec{\beta}}$ as $\frac{1-e^{-\Sigma_p}}{\Sigma_p}\beta(1)p_1$, where the term on the left is decreasing on p_2, and the one on the right is constant.
2. Observe that $l_1(p) = \phi_1\frac{1-e^{-\Sigma_p}}{\Sigma_p}G^3_{\alpha(1)}\circ f^{(1)}_{\vec{\beta}}(p)$. From the analysis above, $f^{(1)}_{\vec{\beta}}$ is increasing and G_α is decreasing, so l_1 is a product of decreasing functions on p_1.
3. If $\phi_1 < 2\log 2$, then from Proposition II.5.1.2 we know that $g'_{\alpha(i)}\circ f^{(i)}_{\vec{\beta}}(p)\geq 0$, so h^1 satisfies the same monotonicity as $f^{(1)}_{\vec{\beta}}$ on each argument. Since $l_1(p) = \frac{h^1(p)}{p_1}$, it must behave as h^1 with respect to p_2.

II.D Proof of Proposition II.5.1.4

Recall the definition of Σ_p used in the proof of Proposition II.5.1.2.

- *Proof of* (II.38)*:* Take c small (to be fixed later), and suppose that $p_2 < c$. Observing that $f^{(2)}_{\vec{\beta}}(p) = \frac{1-e^{-\Sigma_p}}{\Sigma_p}\beta(2)p_2$ we deduce that $\frac{1-e^{-\beta(1)p_1}}{\beta(1)p_1}\beta(2)p_2 < f^{(2)}_{\vec{\beta}}(p) < \beta(2)p_2$, so from the assumption $p_2 < c$ and the monotonicity of G_α, we deduce

$$l_2(p) = (1-\alpha(2))\frac{f^{(2)}_{\vec{\beta}}(p)}{p_2}G^3_{\alpha(2)}\circ f^{(2)}_{\vec{\beta}}(p) \geq \phi_2\frac{1-e^{-\beta(1)p_1}}{\beta(1)p_1}G^3_{\alpha(2)}(\beta(2)c). \qquad \text{(II.64)}$$

  Since the fraction is decreasing on p_1 we obtain a lower bound by taking $p_1 = \kappa_\varepsilon$ and using its definition to obtain

$$l_2 \;\geq\; \phi_2\frac{1-e^{-\beta(1)\kappa_\varepsilon}}{\beta(1)\kappa_\varepsilon}G^3_{\alpha(2)}(\beta(2)c) \;\geq (1-\varepsilon)\frac{\phi_2}{\phi_1}\frac{G^3_{\alpha(2)}(\beta(2)c)}{G^3_{\alpha(1)}(1-e^{-\beta(1)\kappa_\varepsilon})}.$$

  But $\frac{\phi_2}{\phi_1} > 1$ and as $c\to 0$ we have $G_{\alpha(2)}(\beta(2)c)\to 1$, so taking first ε small and then c sufficiently small, the right hand side is larger than $1+\varepsilon'$ for some ε'.

- *Proof of* (II.39)*:* For the second part, Proposition II.5.1.2 gives that $g_{\alpha(2)}$ has a single critical point which is a maximum, so $h_2 = g_{\alpha(2)}\circ f^{(2)}_{\vec{\beta}}$ is minimized either when $f^{(2)}_{\vec{\beta}}$ is minimized or maximized. Remembering that $f^{(2)}_{\vec{\beta}}$ decreases with p_1 and increases with p_2, we conclude that the minimum of h_2 over the set $[0,\kappa_\varepsilon]\times[c,\bar{p}_2]$ is obtained either at $(0,\frac{1-\alpha(2)}{2})$ or at (κ_ε, c). We already saw that at the point $p = (\kappa_\varepsilon, c)$ we have $h_2(p) = l_2(p)p_2 > (1+\varepsilon')c$, meaning that we need only to control h_2 at $(0,\frac{1-\alpha(2)}{2})$, which is equal to $g_{\alpha(2)}(1-e^{-\phi_2/2})$, which is a fixed value so the result follows by taking c small enough so that $g_{\alpha(2)}(1-e^{-\phi_2/2}) > (1+\varepsilon')c$.

- *Proof of Proposition II.5.1.4.(ii):* To prove the properties above for a general value of p_1 we proceed analogously, but when computing (II.64) we use the additional information $\phi_1 < 2\log 2$ to improve the lower bound without imposing any restriction on p_1. Indeed, since $\phi_1 < \phi_2$ we deduce that $\beta(1)p_1 \leq \frac{\phi_2}{2}$ so, from monotonicity of $\frac{1-e^{-x}}{x}$,

$$l_2(p) \geq \phi_2\frac{1-e^{-\beta(1)p_1}}{\beta(1)p_1}G^3_{\alpha(2)}(\beta(2)c) \geq 2(1-e^{-\phi_2/2})G^3_{\alpha(2)}(\beta(2)c),$$

but $2(1-e^{-\phi_2/2}) > 1$ from our assumption $\phi_2 > 2\log 2$, so taking c sufficiently small we conclude again that $l_2(p) > 1+\varepsilon'$ for some ε' small. The proof of the second property is exactly the same as in II.39.

- *Proof of* (II.40) *and* (II.41)*:* Notice that, since $\phi_1 < 2\log 2$, from Proposition II.5.1.3 we know that h_1 is increasing in p_1 and decreasing in p_2, so using the definition of κ_ε we deduce

$$p_1 < \kappa_\varepsilon \implies h_1(p) \le h_1(\kappa_\varepsilon, 0) = g_{\alpha(1)}(1-e^{-\beta(1)\kappa_\varepsilon}) = (1-\varepsilon)\kappa_\varepsilon,$$

which proves (II.40). To prove (II.41) we use a similar argument with l_1, which we know is decreasing in both arguments, so that

$$\kappa_\varepsilon < p_1 \implies l_1(p) \le l_1(\kappa_\varepsilon, 0) = \frac{g_{\alpha(1)}(1-e^{-\beta(1)\kappa_\varepsilon})}{\kappa_\varepsilon} = (1-\varepsilon),$$

and the result follows.

III

Tree-decorated planar maps: counting results.

Contents

We introduce the set of (non-spanning) tree-decorated planar maps, and show an explicit bijection between this set and the Cartesian product between the set of trees and the set of maps with a simple boundary. As a consequence, we count the number of tree decorated triangulations and quadrangulations with a given amount of faces and for a given size of the tree. Finally, we generalize the bijection to study other types of decorated planar maps and obtain explicit counting formulas for them.

III.1 Introduction

In this chapter, we study the combinatorial properties of tree-decorated maps via the use of a simple bijection. A tree-decorated map is a pair made of a rooted (planar) map and a submap that

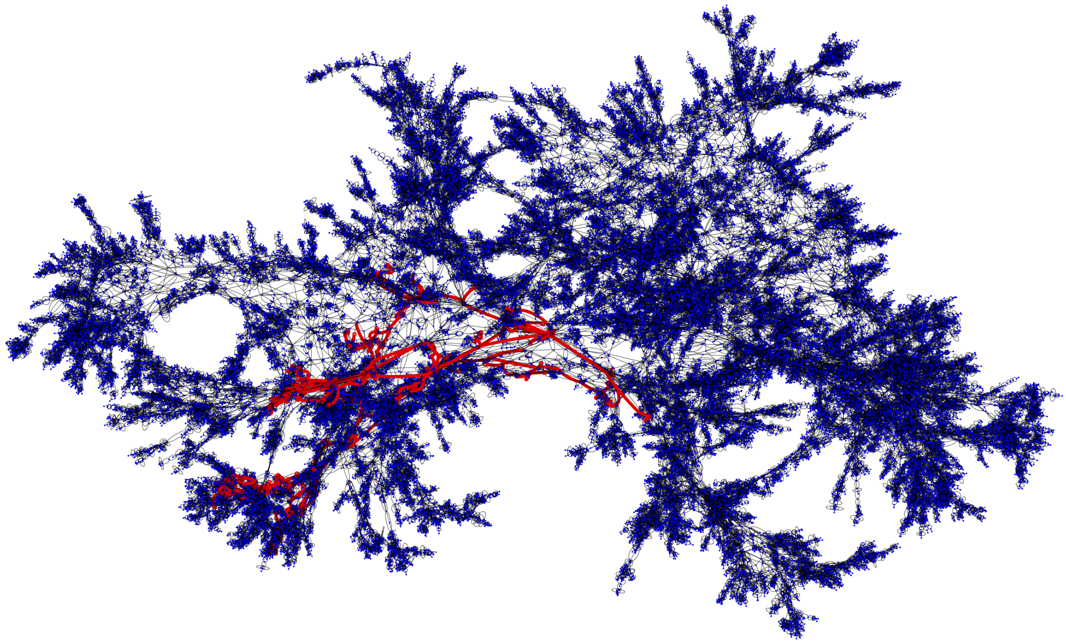


Figure III.1 – A simulation, based on our bijection, of a uniformly chosen tree-decorated quadrangulation among those with 90000 faces and with a tree of 500 edges. The tree is in red.

is a tree. When fixing the tree size as a parameter, tree decorated maps interpolates between planar quadrangulations, when the tree has one edge, and the spanning-tree decorated quadrangulations, when the tree has the same number of vertices that the whole graph.

Planar maps and spanning-tree decorated maps have been studied, both in combinatorics and probability. Planar maps were introduced in [43] and afterwards have been thoroughly studied in many works from both a combinatorial (see for example [113, 100, 20]) and a probabilistic perspective (see for example [87, 77, 89]). Spanning-tree decorated maps were first studied in [93], where a simple counting formula is given, which later was explained in [114, 26, 11] through bijective methods. These bijections are the key to the study of planar maps decorated by statistical physics models [103].

III.1.1 Motivation

The main motivation for the introduction of this model is to try to understand the difference, as metric spaces, between uniformly chosen planar maps and uniformly chosen spanning-tree decorated planar maps.

The study of planar maps as metric spaces has been an active area of research in combinatorics and probability theory these last years. Cori-Vauquelin-Schaeffer (CVS) bijection [100] has been used to understand the case of a uniformly chosen planar quadrangulation with f faces. Many of its asymptotic properties: the distances scale like the number of faces f to the power $1/4$ and there is an explicit limiting metric space called Brownian map [87, 89, 77]. However, in the world of uniformly chosen spanning-tree decorated planar maps (general with a fixed number of edges or q-angulations), we do not know much. Only bounds on the order of the diameter as a function of the number of edges are known [61, 34]. On the optimistic side, Walsh and Lehman's bijection [114] shows that in the case of uniform spanning tree decorated map, the tree is uniformly distributed over the set of all trees with the right number of edges.

The main reason why it is difficult to understand distances in the context of spanning-tree decorated maps becomes clear when one compares it to the case of planar quadrangulations. The main tool used to study distances in these planar maps is the (CVS) bijection, which relates a planar map to a pair of trees. In this bijection, one of the trees encodes the distance to a marked point, thus by using scale order of this tree, one gets an immediate lower and upper bound on the diameter of the map. On the side of tree-decorated planar maps, we are not that lucky. Even though Walsh and Lehman's bijection [114, 11] relate them to a pair of trees, it seems not easy to extract any information about distances in the original graph from them.

At this point, let us make a remark from the point of view of conformal field theory, where these two models are not expected to look the same. Uniformly chosen planar maps are model associated to central charge equal to 0 ($c = 0$), while spanning-tree decorated maps have an associated central charge of -2 ($c = -2$) [68]. Thus, two objects, in the world of tree-decorated maps, have two-different central charges, which gives evidence that theirs conformal properties should change in general with respect to the number of edges of the tree. Trying to understand how this interaction works in the limit is the main interest of a work to come [51].

III.1.2 Results

A rooted planar map is a pair $(\mathfrak{m}, \vec{e})$ made of a map $\mathfrak{m}$, which is an embedding of a finite connected planar graph in the plane (or the sphere $\mathbf{S}^2$), without edge crossings, and an oriented edge $\vec{e}$ (the root-edge), considered up to direct homeomorphisms of the sphere. A map (omit the colors for the moment) is shown in Figure 6, where its root edge is represented by an arrow.

The degree of a face is the number of edges adjacent to it (an edge included in a face is counted twice). A q-angulation is a map whose faces have degree q (Figure 6 shows a 4-angulation; these are also called quadrangulations).

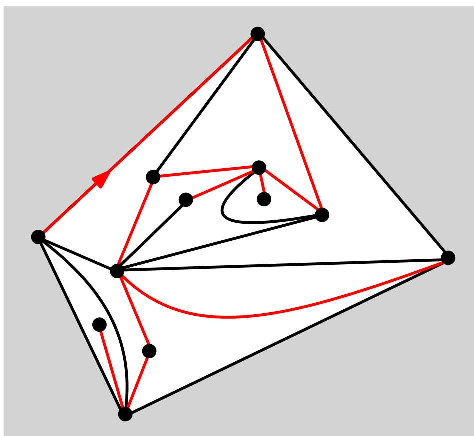


Figure III.2 – A map decorated by a spanning tree.

A rooted plane tree, or tree for short, is a rooted map with one face.

The face that is at the left of the root-edge will be called the root-face or external face (face in gray in Figure 6). In what follows maps with a boundary are maps where the root face plays a special role; and the set of edges that are adjacent to it form the boundary. The number of edges in the boundary will be called its size. The boundary will be seen sometimes as a path around the root face, up to cyclic rotation of the indices, or as a set of edges. All others faces are called internal faces. For example, a quadrangulation with a boundary of size p is a map where all internal faces have degree 4 and the root-face has degree p. The boundary of a map is said to be simple if it forms a non vertex-intersecting path.

A map (rooted or unrooted) $\mathfrak{m}_1$ is said to be a submap of the rooted map $\mathfrak{m}_2$, if $\mathfrak{m}_1$ can be obtained from $\mathfrak{m}_2$ by suppressing edges and vertices. If the submap $\mathfrak{m}_2$ contains the root of $\mathfrak{m}_1$, then it is rooted, otherwise it is unrooted.

Definition III.1.2.1
For $(f, a) \in (\mathbb{N}^*)^2$ a **rooted** (f, a) **tree-decorated map** is a triplet $(\mathfrak{m}, \mathfrak{t}, \vec{e})$ where $(\mathfrak{m}, \vec{e})$ is a rooted map with f faces, and $(\mathfrak{t}, \vec{e})$ is a rooted plane tree with a edges, which is a submap of $\mathfrak{m}$.

By definition, in a rooted tree-decorated map the root-edge of the decoration and the root-edge of the map coincide (this will not be the case further for tree-decorated maps).

For other detailed definitions, we refer the reader to Section III.2. Let us, now, present the main results of the chapter.

The corner stone of this chapter is an explicit bijection between the set of tree-decorated maps and the set of pairs made by a rooted maps with a simple boundary and a rooted tree.

Theorem III.1.2.2
There exists an explicit bijection g between:

- the set of rooted (f, a) tree-decorated maps and
- the Cartesian product between the set of rooted trees with a edges and the set of rooted maps with a simple boundary of size $2a$ (boundary with $2a$ edges) and f interior faces.

The bijection can be summarised as follows: a copy of the tree is kept aside and the map with a simple boundary is generated by inflating the tree decoration, this latter becoming in this way a face with a simple boundary of size twice the size of the tree (see Figure III.3). We call this direction of the bijection ungluing procedure/function u, and its inverse the gluing procedure/function g. The gluing function g consists only in identifying the boundary of the map using the equivalence relationship generated by the tree.

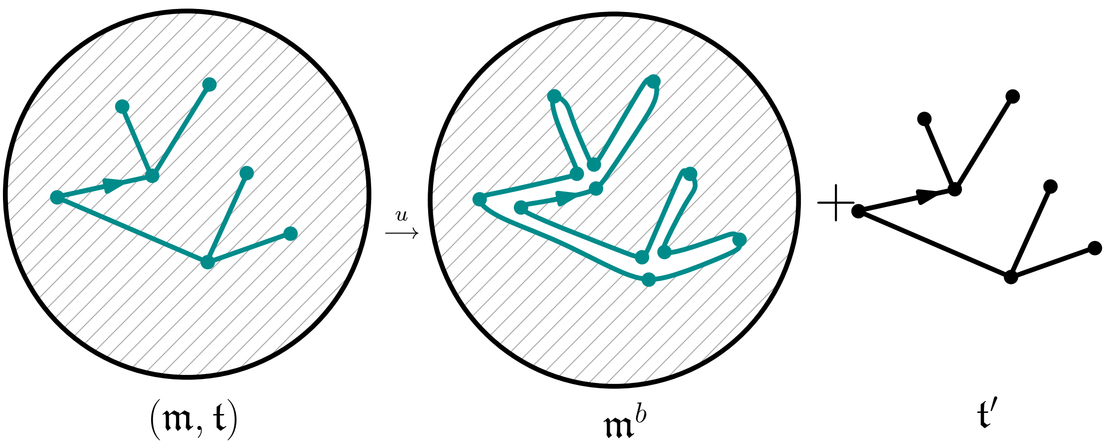


Figure III.3 – A simple sketch of the ungluing procedure.

In view of the motivation, the main Theorem III.1.2.2 allows us to connect the study of tree-decorated maps with maps with a boundary. Maps with boundary are also an interpolation model, as when the boundary is of size 2, they coincide with planar maps, and, when the boundary has twice the number of edges, we recover plane trees. This interpolation effect was explored, for uniformly chosen maps (with a given size) with a boundary in [13], and we hope g will allow us to transfer the transition from maps to trees, hold in the case of maps with a boundary, to that from maps to spanning tree-decorated maps, in tree-decorated maps.

Definition III.1.2.3
Fix $(f, a) \in (\mathbb{N}^*)^2$. A (f, a) **tree-decorated map** is a triplet $(\mathfrak{m}, \mathfrak{t}, \vec{e})$ where $(\mathfrak{m}, \vec{e})$ is a rooted map with f faces, and $\mathfrak{t}$ is a tree with a edges, submap of $\mathfrak{m}$.

As a submap, the tree $\mathfrak{t}$ can be rooted by $\vec{e}$ or unrooted, meaning that it could inherit or not the root-edge $\vec{e}$ of the map $\mathfrak{m}$.

This "possibly unrooted version" of rooted tree-decorated maps is introduced in order to fit with the literature, since for example, in the case of spanning-tree decorated maps the root-edge of the map is not necessarily on the tree.

Important consequences of Theorem III.1.2.2 are counting formulas for some subsets of tree-decorated maps. A close look at the ungluing procedure u shows that it only creates a new face, so that it does not modify the internal faces of the map. Thus, it is possible to obtain counting formulae for tree-decorated (and spanning-tree decorated) q-angulations. To obtain these results, we need to count the maps with a simple boundary and use a re-rooting argument (this is explained in Section III.4.1). Luckily, we can find these countings in [73] for triangulations and in [19] for quadrangulations.

Corollary III.1.2.4
For $a \leq f/2 + 1$, the number of (f, a) tree-decorated triangulations is

$$2^{f-2a} \frac{(3f/2 + a - 2)!!}{(f/2 - a + 1)!(f/2 + 3a)!!} \frac{3f}{a+1} \binom{4a}{2a, a, a}, \tag{III.1}$$

where $n!!$ stands for the double factorial of n.

Furthermore, for $a \leq f + 1$, the amount of (f, a) tree-decorated quadrangulations is

$$3^{f-a} \frac{(2f + a - 1)!}{(f + 2a)!(f - a + 1)!} \frac{4f}{a+1} \binom{3a}{a, a, a}. \tag{III.2}$$

Additional note.

> Both bounds $f/2 + 1$ and $f + 1$ corresponds to the number of edges in a spanning trees for triangulations and quadrangulations with f faces, respectively. To get these numbers, one can compute the number of vertices in triangulations and quadrangulations with f faces, thanks to the Euler formula, and double counting arguments.

The formulas for spanning-tree decorated quadrangulations and spanning-tree decorated triangulations appear in [18], and are based in Walsh and Lehman's bijection [114]. Our approach will provide a new bijective proof of them.

Corollary III.1.2.5
The number of spanning-tree decorated triangulations with f faces, i.e. $(f, f/2+1)$ tree-decorated triangulations, is

$$\frac{12f}{(f+4)(f+2)^2}\binom{2f}{f, f/2, f/2}. \tag{III.3}$$

Furthermore, the number of spanning-tree decorated quadrangulations with f faces, i.e. $(f, f+1)$ tree-decorated quadrangulations, is

$$\frac{4f}{(f+1)^2(f+2)}\binom{3f}{f, f, f}. \tag{III.4}$$

Remark III.1.2.6
Additionally, let us note that as the decorating tree is kept without any changes, when one picks a uniform rooted (f, a) tree-decorated q-angulation (or maps with prescribed number of total edges), the law of the tree is uniform among all the rooted trees with a edges. This is a generalization of the result for spanning-trees decorated maps. Furthermore, the power of this bijection is that it can be restricted to different families of maps (q-angulations) and tree decorations (d-regular trees).

The counting formula for general maps with some given boundary and edges sizes is given through its generating function in Section III.4.2.

The bijections u and g "do not change" the vertices and edges that are not adjacent to the tree (or to the boundary for g) so the bijections induces a correspondence between maps with a given number of total edges.

Definition III.1.2.7
Fix $(e, a) \in (\mathbb{N}^*)^2$. A **rooted** $[e, a]$ **tree-decorated map** is a triplet $(\mathfrak{m}, \mathfrak{t}, \vec{e})$ where $(\mathfrak{m}, \vec{e})$ is a rooted map with e edges, and $(\mathfrak{t}, \vec{e})$ is a rooted tree with a edges, submap of $\mathfrak{m}$.

Theorem III.1.2.8
There exists a bijection g_e between:

- the set of rooted $[e+a, a]$ tree-decorated maps and
- the Cartesian product of rooted trees with a edges and rooted maps with a simple boundary of size $2a$ and e interior edges.

The bijection is not only useful to obtain counting results, but we plan to use it in [51] to transfer asymptotic results concerning maps with a boundary, intensively studied in [13, 14, 58] and well understood, to tree-decorated maps.

Let us also mention that the bijection presented in this work is simple enough so that attributes on the faces can be carried between the two objects. This may allow in the future to understand tree-decorated maps weighted by a statistical physic law. This probability law, have been the object of great interest in the statistical physics community, especially after the introduction of the so-called 'Hamburger-Cheeseburger'-bijection [103].

Most of the other results of this chapter consist in using the main idea of the bijection of Theorem III.1.2.2 to produce bijections between other combinatorial objects. We study forest-decorated planar maps (see Corollary III.4.3.2) and tree-decorated planar maps with a simple boundary (see Corollary III.4.3.1). The latter allows us to make the gluing procedure in a progressive dynamical way (see Proposition III.3.3.1).

In Section III.3.3, we also explain what happens when "one tries to apply our gluing function g" to maps whose boundary is not necessarily simple. In this case, the natural result is not a maps, but what we call bubble-maps (see Section III.3.3), which are maps embedded in a tree-like structure of spheres, and which are decorated by a specific type of circuit (see Proposition III.3.3.1).

As a final remark, we would like to say that gluing maps, by folding the boundary or by gluing some boundaries, has already been considered, but not in relation with tree-decorated maps, for example in relation with maps decorated by self avoiding walks [39, 13, 57, 22] and loops [17].

III.1.3 Organisation of the chapter

The chapter is organized as follows: we start by adding some elements on the combinatorial objects we are interested in. In Section III.3, we present all the bijections and theirs proofs. Finally, in Section III.4, we discuss the counting formulas we obtain from the bijections.

III.2 Preliminaries

III.2.1 Elementary definitions

For an introduction to planar maps, we recommend, for example, to see [54, 48] and [18]. For the definitions of maps, faces, trees, boundary, submaps and others we refer to the Section III.1.2.

According to the context, we will sometimes consider the edges of maps as two directed edges (the two possible directions), two half edges or one (non-directed) edge. We call **root-vertex** the starting point of the root-edge. We associate to each face the set of oriented edges having the face to its left.

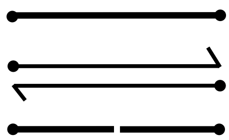


Figure III.4 – Representations of an edge.

Additional note.

> There are several equivalent ways to root a map: the one we have chosen "the root-edge" is equivalent to the choice of "marked corner" or to the choice of a "half-edge".

We call **rooted plane tree** a rooted map with a single face and we denote the set of rooted trees with a edges by T_a. The number of edges in a tree will be called the size of the tree.

Additional note.

By definition a rooted plane tree is a map with a single face, therefore, connected and acyclic (any cycle, in the planar setting, induces more than one face). Moreover, since it is embedded on the sphere, there is a special rotation order around the tree, which consists in following the boundary of its unique face from the root-edge. This visiting order coincides with what we call later, the contour function.

We will encode trees of T_a using walks. In the literature, one can find several of these codings, see for example Section 1 of [75]. In this chapter, we are interested in: the contour function, bijection that associates to each rooted plane tree with a edges a Dyck path C indexed by $[\![0, 2a]\!]$. For a more detailed description we refer to Section 1.1 of [75] and Section 2 of [11].

A rooted plane tree $\mathfrak{t}$ has an intrinsic way of visiting all of its oriented-edges [1]. This visit can be represented by a walker that starts from the root vertex and turns around the tree: he follows the direction of the root edge touching the tree with his left hand [2] as long as it walks. The walker, then, continues until he returns to the root edge. Note that this walk visits every oriented edge only once. Now, we define the contour function $C : [\![0, 2a]\!] \to \mathbb{N}$ as the function (we should write $C_{\mathfrak{t}}$ instead of C) for which $C(n)$ is the distance to the root vertex (height) of the vertex visited at time n by the walker (time 0 for the root-vertex).

The inverse of this bijection $\mathfrak{t} \to C_{\mathfrak{t}}$ is explicit. We say that a function $C : [\![0, 2a]\!] \to \mathbb{N}$ is a contour function if $C(0) = C(2a) = 0$ and its increments are ± 1, i.e., C is a Dyck path. We can construct a plane tree from a contour function by saying that two points $n_1, n_2 \in [\![0, 2a]\!]$ are equivalent if for $n_1 \leq n_2$

$$C(n_1) = C(n_2) = \inf_{n \in [\![n_1, n_2]\!]} C(n). \tag{III.5}$$

The vertices of the tree are the equivalence classes of the relation and the edges can be recovered as follows: two vertices have an edge between them if they are the equivalence classes of two elements that are exactly at distance 1 in $[\![0, 2a]\!]$. Note that each edge comes exactly from two steps of the walk, one going up and the other one going down.

We define a **non-self crossing circuit** as a sequence of non-crossing directed edges $(e_i)_{i=0}^{l-1}$, for some $l \geq 1$, embedded in the plane, such that the head of e_i is the tail of $e_{i+1 \mod l}$ for all $i \in [\![0, l-1]\!]$. Non self-crossing means that for every vertex x in the circuit, we do not find around x the pattern $e_i, e_j, e_{i+1}, e_{j+1}$ in cyclical (clockwise or anticlockwise) order, where $i, j \in [\![0, l-1]\!]$ and the sum in the indexes is modulo l (see fig. III.6).

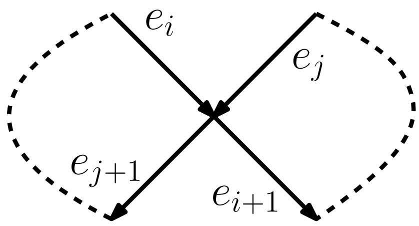


Figure III.6 – Forbidden clockwise circuit.

We denote by $\mathsf{B}_{f,p}$ the set of maps with boundary of size p with f internal faces (see fig. III.7). Note that oriented edges around the boundary of a map have a canonical labeling in $[\![0, p-1]\!] := \{0, 1, \ldots, p-1\}$ coming from the amount of step that a walker, who starts from the root edge and who follows the boundary of the root face, takes to arrive to a given edge (label 0 for the root-edge).

The set of maps with a simple boundary in $\mathsf{B}_{f,p}$ is denoted by $\mathsf{SB}_{f,p}$ (see fig. III.7). When the boundary of the map is simple, i.e., the boundary is not vertex-intersecting, in this case, the labels

1. This is the first time we consider each non-oriented edge as two oriented edges as we remark in Figure III.4.
2. Note that, in the literature, the walker usually walks following its right hand. In this work, the left hand convention makes some statements easier.

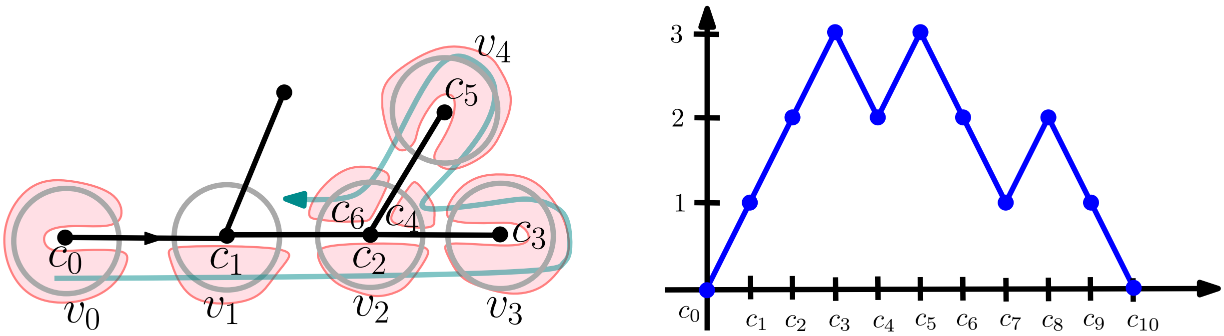


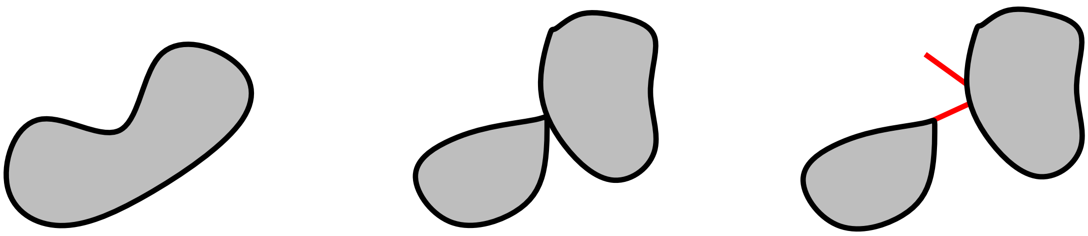
Figure III.5 – Left: Tree with walker path around the tree in cyan, the corners are numbered by the c_i and are the circular sectors shown in red. All the corners belonging to the same gray circle belongs to the same equivalence class associated to a vertex, for example $v_2 = [c_2]_c = [c_4]_c = [c_6]_c$.
Right: Contour function.

of the edges induce a labeling of the vertices of the boundary (the label of the root-vertex is 0).

Additional note.

> In other words, the boundary is said to be simple iff, when one turns around the boundary each vertex adjacent to the boundary is visited once.

Let us also introduce another type of boundary. A boundary is called **bridgeless** if the walk described above never goes twice along the same edge. Let us give an alternative definition. An edge is called a bridge if its suppression disconnects the map. Thus, a boundary is said to be bridgeless if it does not contain any bridges (see fig. III.7).



(a) Simple bridgeless boundary. (b) Non-simple bridgeless boundary. (c) Non-simple boundary with bridges (in red).

Figure III.7 – Different types of boundary, the gray region is the interior of the map and the unbounded region is the outer face.

A rooted decorated map is a map with a special submap, i.e. it is a triplet $(\mathfrak{m}, \mathfrak{sm}, \vec{e})$ with $(\mathfrak{m}, \vec{e})$ a rooted map with f faces and $\mathfrak{sm} \subset_M \mathfrak{m}$.

III.2.2 Tree-decorated maps

Let $\mathring{\mathsf{M}}_f^{\mathsf{T},a}$ be the set of all rooted (f, a) tree-decorated maps.

Definition III.2.2.1
Fix $a, f, p \in \mathbb{N}^*$. A (f, p, a) **tree-decorated map with a simple boundary** is a triplet $(\mathfrak{m}^b, \mathfrak{t}, \vec{e})$ with $\mathfrak{m}^b$ a rooted map with a simple boundary in $\mathsf{SB}_{f,p}$, rooted on $\vec{e}$ and $\mathfrak{t}$ a submap of $\mathfrak{m}^b$, which is a tree with a edges, intersecting the boundary only at the root-vertex.

We denote by $\mathsf{B}_{f,p}^{\mathsf{T},a}$ the set of all (f, p, a) tree-decorated map with a simple boundary. The cardinality of the set of all tree-decorated triangulations with a simple boundary is counted by (III.15) and that of quadrangulations Corollary III.1.2.5 is counted by (III.16). We set

$$\mathsf{B}_{f,0}^{\mathsf{T},a} := \mathring{\mathsf{M}}_f^{\mathsf{T},a},$$

when the parameter $p = 0$.

III.3 Main bijections

III.3.1 The basic bijection

From rooted tree-decorated maps to rooted maps with simple boundary and rooted trees:

In the following paragraphs, we define formally the ungluing function u sketched in Figure III.3. The function u takes as argument a rooted tree-decorated map in $\mathring{\mathsf{M}}_f^{\mathsf{T},a}$, its image is a rooted tree in T_a together with a rooted map with a simple boundary in $\mathsf{SB}_{f,2a}$. Basically, the resulting tree is equal to the decorating tree and the map with a boundary is obtained by a duplication of the oriented edges of the tree in such a way that the newly appeared oriented edges form a face, see Figure III.3.

Consider a tree decorated map $(\mathfrak{m}, \mathfrak{t}, \vec{e}) \in \mathring{\mathsf{M}}_f^{\mathsf{T},a}$, and denote $((\mathfrak{m}^b, \vec{e}_b), (\mathfrak{t}', \vec{e}_{t'}))$ the (soon-to-be-constructed) image of $(\mathfrak{m}, \mathfrak{t}, \vec{e})$ under u. The tree $(\mathfrak{t}', \vec{e}_t)$ is taken equal to the tree $(\mathfrak{t}, \vec{e})$, in particular the root edge of $\mathfrak{t}'$ is the same as that of $\mathfrak{t}$, i.e., the one of the map $\mathfrak{m}$.

To construct the map with a simple boundary $(\mathfrak{m}^b, \vec{e}_b)$ we start by defining the notion of a corner of $\mathfrak{t}$ at a vertex x as a pair of two consecutive oriented edges (for the clockwise order), where the first one finishes at x and the second one starts at x. Define K as the set of corners of $\mathfrak{t}$. An oriented edge that starts or ends at x is said to go to a corner (e_1, e_2) of x if it is between e_1 and e_2 for the clockwise order.

We can now define $(\mathfrak{m}^b, \vec{e}_b)$. The set of vertices of $\mathfrak{m}^b$ is the union between K and the subset of vertices of $\mathfrak{m}$ that are not in the tree $\mathfrak{t}$, i.e., $V(\mathfrak{m}) \backslash V(\mathfrak{t})$. To define the edges of $\mathfrak{m}^b$ we will use those of $\mathfrak{m}$. Each oriented edge $\vec{e}\,'$ of $\mathfrak{m}$ whose vertices do not lie in $\mathfrak{t}$ is also an oriented edge of $\mathfrak{m}^b$. Each oriented edge $\vec{e}\,'$ of $\mathfrak{m}$ that has at least one vertex in $\mathfrak{t}$, becomes an oriented edge that instead of the vertex in the tree has the corner in which the edge is incident as extremity. Finally, we add the oriented edges (c_1, c_2) between two elements of K if c_1 and c_2 seen as corners in $\mathfrak{t}$ are connected by an oriented edge in $\mathfrak{t}$ (see fig. III.8).

To finish the definition of $\mathfrak{m}^b$ we are only missing the root-edge and the embedding. The root-edge $\vec{e}_b$ is the image of $\vec{e}$ from the edge definition described above. The embedding is characterized by the cyclic orientation of the edges which is the same as that of $\mathfrak{m}$. In other words, in a corner the cyclic orientation is kept the same as in $\mathfrak{m}$, and this can be done because all edges in $\mathfrak{m}^b$ that belong to a corner can be identified with a unique edge in $\mathfrak{m}$.

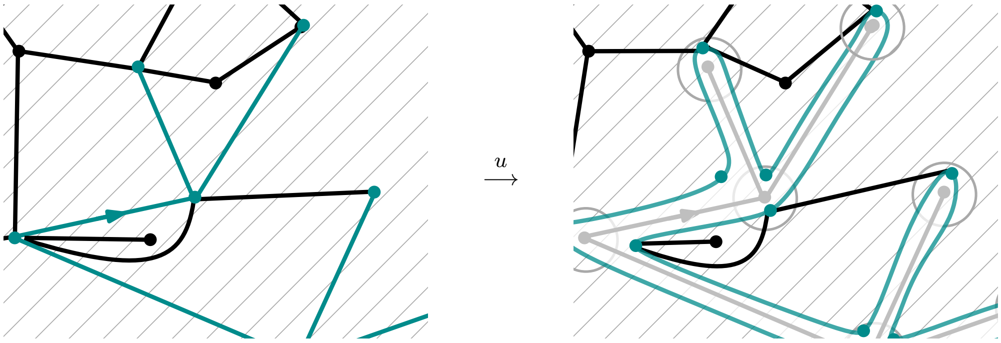


Figure III.8 – Local image of u

Let us, now, show the following:

Lemma III.3.1.1
The constructed map $(\mathfrak{m}^b, \vec{e}_b)$ belongs to $\mathsf{SB}_{f,2a}$.

Proof. First the boundary has clearly size $2a$, since the edges of $\mathfrak{t}$ have been duplicated. Furthermore, the boundary of the map $\mathfrak{m}^b$ is simple, otherwise the tree would contain a cycle.

To finish, we just need to show that $\mathfrak{m}^b$ is a (planar) map, that is, it can be embedded in the sphere $\mathbf{S}^2$. This comes directly from the fact that there is an orientation preserving isomorphism that goes from $\mathbf{S}^2 \backslash \mathfrak{t}$ to $\mathbf{S}^2 \cap \{(x,y,z) : x > 0\}$, and that can be extended to the boundary in a way that it sends prime ends of $\mathfrak{t}$ in $\mathbf{S}^2 \backslash \mathfrak{t}$ to points in $\mathbf{S}^2 \cap \{x = 0\}$ keeping the cyclical order between them [3]. Thus, one can embed $\mathfrak{m}^b$ in $\mathbf{S}^2$ using any of these functions. To finish, it is enough to see that any of these isomorphisms produce an embedding of $\mathfrak{m}^b$ with boundary contained in $\mathbf{S}^2 \cap \{x = 0\}$. □

Finally, let us note that $\mathfrak{m}^b$ has exactly one more face than $\mathfrak{m}$: the new root-face. Furthermore, the internal faces of $\mathfrak{m}^b$ are in one to one correspondence with the faces of $\mathfrak{m}$. Thus, we can conclude the following lemma.

Lemma III.3.1.2
For any $q \in \mathbb{N}$, the ungluing function u sends the set of rooted (f,a) tree-decorated q-angulation $(\mathfrak{m}, \mathfrak{t}, \vec{e})$ to the set of pairs $((\mathfrak{m}^b, \vec{e}_b), (\mathfrak{t}', \vec{e}_t))$ where $(\mathfrak{m}^b, \vec{e}_b)$ is a rooted q-angulation with boundary of size $2a$ and $(\mathfrak{t}', \vec{e}_t)$ is a rooted tree of size a.

From maps with simple boundary and trees to tree decorated maps

It is time to define formally the gluing function g that takes as argument a map with a boundary in $\mathsf{SB}_{f,2a}$ and a tree in T_a and returns as result a rooted (f,a) tree-decorated map. It will be proved that it is the inverse of u. Informally, g should identify two oriented edges incident to the external face of the map $\mathfrak{m} \in \mathsf{SB}_{f,2a}$ using the relation given by the oriented edges in the tree.

3. In this case it follows from the fact that $\mathbf{S}^2 \backslash \mathfrak{t}$ is simply connected (so one can use Riemman's theorem), together with results of the behaviour of conformal functions close to the boundary. See Chapter 2 of [96] for definitions of prime ends and the main results in the boundary behaviour of conformal maps.

Let $((\mathfrak{m}^b, \vec{e}_b), (\mathfrak{t}', \vec{e}_t))$ be an element of $\mathsf{SB}_{f,2a} \times \mathsf{T}_a$, we will construct $(\mathfrak{m}, \mathfrak{t}, \vec{e}) \in \mathring{\mathsf{M}}_f^{\mathsf{T},a}$, the value of $g((\mathfrak{m}^b, \vec{e}_b), (\mathfrak{t}', \vec{e}_t)))$ as follows. Recall that the vertices of the external face of $\mathfrak{m}^b$ are indexed from 0 to $2a-1$, and call C the contour function of $\mathfrak{t}'$. Recall that C induces an equivalence relation on the set of corners via equation (III.5), and define V' as the set of equivalence classes.

Let us now construct $\mathfrak{m}$. The vertex set of $\mathfrak{m}$ is made by the union of V' with the set of vertices of $\mathfrak{m}^b$ that do not belong to the exterior face. The edge set of $\mathfrak{m}$ is constructed from that of $\mathfrak{m}^b$ in the following way. Let (x, y) be an oriented edge of $\mathfrak{m}^b$, then the edge $(G(x), G(y))$ is in $\mathfrak{m}^b$, where

$$G(x) := \begin{cases} [l] & \text{if } x \in V', \\ x & \text{else}, \end{cases} \tag{III.6}$$

where l is the label of x in the boundary (from the simple boundary behavior commented in the introduction), and $[l]$ is the equivalence class of l under the equivalence relation defined by C.

Before defining the embedding, let us give some properties of the resulting graph.

- If (x, y) is an oriented edge that belongs to the boundary of $\mathfrak{m}^b$, its reverse (y, x) is associated exactly to one other edge (y', x') such that (x', y') belongs to the boundary of $\mathfrak{m}^b$. In other words, C induces a perfect matching of the edges in the boundary
- As the boundary of $\mathfrak{m}^b$ is simple, the image of the edges in the boundary of $\mathfrak{m}^b$ has the same tree structure as $\mathfrak{t}'$. We define $\mathfrak{t}$ as this image.

We define $\vec{e}$ as the image of $\vec{e}_b$ under the assignation of edges. Notice that the identification satisfies that $\vec{e}$ is an oriented edge of $\mathfrak{t}$.

To finish the construction of $\mathfrak{m}$, we need to set the cyclical order of the edges around each vertex. If v is a vertex of $\mathfrak{m}$ that does not belong to V', we set the order or the edges surrounding it as its order in $\mathfrak{m}^b$. In the case where $v \in V'$, we consider the order as the gluing of orders, following the corner identification around v (see Figure (III.9)). Note that this creates for a vertex $v \in V(\mathfrak{t})$ and a vertex $\bar{v} \in V(\mathfrak{t}')$, where v is the image of $\bar{v}$ for the gluing g, the same cyclical order.

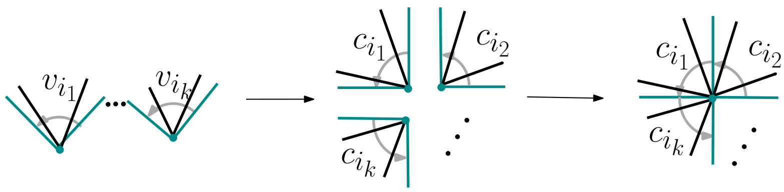


Figure III.9 – Gluing of corner and edge orders around a vertex. For vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ on the boundary of $\mathfrak{m}^b$ respecting the cyclic order around the boundary, to identify these vertices by the gluing procedure, we do it roughly as the right side picture. Choose a point x of the plane and partition the plane around x into k angular sectors, put v_{i_j} at x and draw its incident edges in the j-th angular sector, respecting the order.

To finish, let us actually prove that there exists an embedding that satisfy the given properties.

Lemma III.3.1.3
$(\mathfrak{m}, \mathfrak{t}, \vec{e})$ (as constructed above) belongs to $\mathring{\mathsf{M}}_f^{\mathsf{T},a}$.

Proof. We already know that $\mathfrak{t}$ is a submap of $\mathfrak{m}$ and that it is a tree with a edges, we just need to prove that there exists an embedding that satisfies the order we imposed. This follows, like in Lemma

III.3.1.1, the existence of an orientation preserving isomorphism that goes from $\mathbf{S}^2 \cap \{(x,y,z) : x > 0\}$ to $\mathbf{S}^2 \backslash \mathfrak{t}$, and that takes any cyclically ordered prime ends of t in $\mathbf{S}^2 \backslash \mathfrak{t}$ to elements of $\mathbf{S}^2 \cap \{x = 0\}$ keeping the same cyclical order. □

Remark III.3.1.4
Consider T a subset of T_a the set of rooted trees with a edges. The gluing procedure g can be restricted to the family $\mathsf{SB}_{f,2a} \times T$ and with image set corresponding to tree-decorated map for which the decoration belongs to T.

The gluing is actually a bijection

Let us now prove that the gluing and ungluing functions g and u are inverse of each other.

Proposition III.3.1.5
For $f, a \in \mathbb{N}^*$, $g \circ u = Id$, the identity in $\mathring{\mathsf{M}}_f^{\mathsf{T},a}$ and $u \circ g = Id$, the identity in $\mathsf{SB}_{f,2a} \times \mathsf{T}_a$.

Proof. Consider $(\mathfrak{m}, \mathfrak{t}, \vec{e}) \in \mathring{\mathsf{M}}_f^{\mathsf{T},a}$, and let us show that $g(u((\mathfrak{m}, \mathfrak{t}, \vec{e}))) = (\mathfrak{m}, \mathfrak{t}, \vec{e})$. First note that the composition of both functions preserves the number of vertices of $\mathfrak{m}$. This is because, every vertex that does not belong to the decoration does not change by the transformations and the vertices on the decoration are separated by u in corners of $\mathfrak{t}$, and then gathered by g in exactly the same vertices of $\mathfrak{m}$. Let us now note that the edges are kept the same by the composition $g \circ u$. This is because every edge without endpoints in the decoration are unchanged by g and by u, and edges with endpoints in the decoration again are unglued from the decoration by u and glued back by g.

Finally, the cyclical orders of edges around the vertices are kept: the reason is that for vertices that are not in the tree the cyclical order is kept the same, and for edges that intersect the tree, this order is preserved since u inflates the tree and g deflates the tree.

The proof for $u \circ g = Id$ follows the same lines: the gluing and ungluing functions are designed in order to conserve the local properties. □

Remark III.3.1.6
Proposition III.3.1.5 can also be proven using that the isomorphisms taken in Lemma III.3.1.1 and III.3.1.3 may be chosen to be the inverse of each other.

It is important to remark that the bijection g makes a correspondence between:

Tree-decorated map		**[Map with a simple boundary, tree]**
Faces of degree q	$\longleftrightarrow$	Internal faces of degree q
Internal vertices of degree d	$\longleftrightarrow$	Internal vertices of degree d
Internal edges	$\longleftrightarrow$	Internal edges
Corners of the tree	$\longleftrightarrow$	Boundary vertices.

As already said, Lemma III.3.1.2 together with Proposition III.3.1.5 imply that the gluing-ungluing procedures can be also use to produce a bijection where instead of fixing the number of faces, one fixes the number of edges in the map, which justifies Theorem III.1.2.8. They also imply that g is a bijection when restricted to q-angulations.

Proposition III.3.1.7
The function g is a bijection (more exactly, it induces by restriction a bijection) between :

- the set of rooted (f, a) tree-decorated q-angulations and
- the Cartesian product between the set of rooted trees with a edges and the set of rooted q-angulations with a simple boundary of size $2a$ and f faces.

Furthermore, as the ungluing keeps the tree without any changes we obtain the following probabilistic result.

Corollary III.3.1.8
The tree of a uniform rooted (f, a) tree-decorated q-angulation or rooted $[e, a]$ tree-decorated maps is uniform in the set of rooted trees of size a.

III.3.2 Extensions

In the following section, we discuss some extensions of the gluing procedure which allows us to make bijections for other tree-decorated map families.

Tree-decorated map with a simple boundary

In this subsection, we study the gluing of trees with maps with a boundary that is bigger than the contour of the tree, and we create a dynamic gluing of the boundary.

Extension of the gluing procedure g to (f, p, a) tree-decorated maps with a simple boundary: Recall the definition of (f, p, a) tree-decorated map with a simple boundary in section III.2.2.

For $a, f, p \in \mathbb{N}^*$, we extend the gluing function g to $\mathsf{SB}_{f,p+2a} \times \mathsf{T}_a \to \mathsf{B}^{\mathsf{T},a}_{f,p}$, meaning that we glue a rooted tree with contour smaller than the boundary of a simple boundary map, the exterior face in this case shrinks but does not disappear. The resulting decorated map has a decoration that is a tree that shares only one vertex with the exterior face. We define the root of the resulting decorated map as the image of the edge labeled $2a$ in the simple boundary map (See **??**). This can be made formal, adapting the proof of theorem III.1.2.2, to conclude the following proposition.

Proposition III.3.2.1
The function g is a bijection (by restriction) between:

- the set of rooted (f, p, a) tree-decorated map with a simple boundary $\mathsf{B}^{\mathsf{T},a}_{f,p}$ and
- the Cartesian product between rooted trees with a edges and rooted maps with a simple boundary with f faces and boundary of size $p + 2a$.

III.3.3 Gluing of trees with non-simple boundary maps

To finish this section, we introduce a gluing procedure for the case when the boundary of the map glued is not simple. The combinatorial objects that appear do not seem canonical, so we discuss with more emphasis the complications which arise.

We mostly focus on the case, where the boundary is bridgeless. In that case, we will make sense of a generalization of the gluing procedure. After that we will explain why the gluing procedure "does not make sense" when the boundary has bridges.

Let us start discussing the gluing between a rooted map with a bridgeless boundary and a rooted tree. Note that, in this case, the resulting glued "map" has a decoration that is not necessarily a tree, but a submap (see Figure III.10). This generates two problems.

- The first one comes from the fact that gluing function is not injective. This is not a central problem as it can be fixed: instead of considering the decoration as a submap, one can consider it as a non-self crossing circuit, defined in Section III.2.1. The circuit is just the image of the contour of the tree under the gluing. The fact that the boundary was bridgeless, implies that the circuit only passes once by each oriented-edge.

-The second problem that arises in the gluing is the main one: there are, in fact, two types of cycles that may appear in the circuit. To describe them, let us first note that cycles can only appear in vertices that come from a pinch point of the boundary. Thus, there are two possibilities for the image under the gluing of these vertices. Either, the gluing identifies the corners of a pinch point with different vertices of the tree. These generate cycles that preserve the topology. On the other hand, if the gluing identifies two corners of a pinch point with the same vertex in the tree, then a 'wicked' point appear. That is to say, this vertex pinches down the sphere, and the graph can no longer be embedded in the sphere but only in a topological space with bubbles.

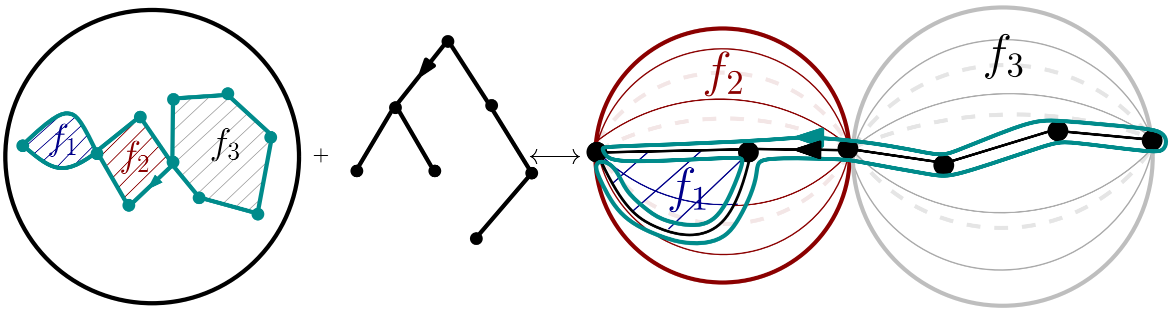


Figure III.10 – Left: Bridgeless map with a non-simple boundary (interior faces are filled) and a tree. Right: Bubbles (3D plot) form by the gluing of a map with non-simple boundary and a tree. We chose to leave the scar generated by the tree after the gluing (black), even though the decoration is the green circuit (oriented edges following the sense of the root edge from the root edge).

Let us explain, in a better way, what happens with the wicked points. For that it is useful to assume that the map with a boundary is already embedded in the sphere. Let x be a pinch point that generates a wicked point in the resulting object. We know that if we remove x, the map with a boundary is left with two or more connected components, say CC_1 and CC_2. Note that two faces, f_1 belonging to CC_1 and f_2 belonging to CC_2 can only be connected through a continuous path in the sphere that either passes through x or through the exterior face of the map. After the gluing has been done, the exterior face disappears. Thus, any continuous path that goes from the image of f_1 to that of f_2 has to pass by x, as the tree does not code the difference between these edges. Thus, *the resulting glued "map", is not a map* since it cannot be embedded in the sphere: removing the point x results in a disconnection of the faces, which never happens in $\mathbf{S}^2$.

Additionally note that "bubbles" are connected in an arborescent way, as they do not form cycles. This is because, each bubble is associated with a starting point and a subtree of the original tree, where two subtrees can only intersect in at most one vertex (See Figure III.10).

Bubble maps definition In the next paragraphs, we will define more rigorously the notion of bubble maps. We need to define the "supporting topological spaces", which are surfaces homeomorphic to $\mathbf{S}^2$ in the case of planar maps. The construction defined previously produces bubbles that can be imagined as on Figure III.11.

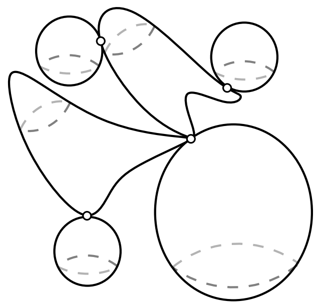


Figure III.11 – Bubble structure.

However, if this can be used for the intuition we need to work at a topological level (for example, we will not assume that "bubbles surfaces" are subsets of $\mathbb{R}^3$). We will rather call an **oriented bubble topological space** (OBTS for short) a space S having the following characteristics:

- S is the union of a finite number b of surfaces, called thereafter "bubbles" $B_1, B_2, \dots, B_b$ taken in a fixed order (meaning that the label i of B_i will play a role, see Figure III.14 for an idea). The sequence of bubbles $(B_1, \cdots, B_b)$ satisfies:
  — Each B_j is homemorphic to $\mathbf{S}^2$ and oriented.
  — The intersection $B_j \cap B_{j'}$ is either empty or a single point, which is a vertex of the graph, for every $j \neq j'$.
  — More than two bubbles can intersect at the same point.
- The global bubble adjacency graph AG (to be defined) is a tree (see Figure III.12). We define

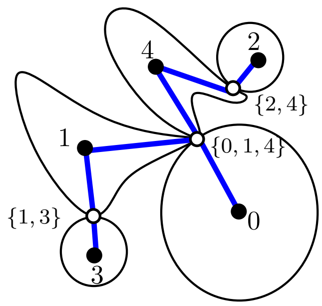
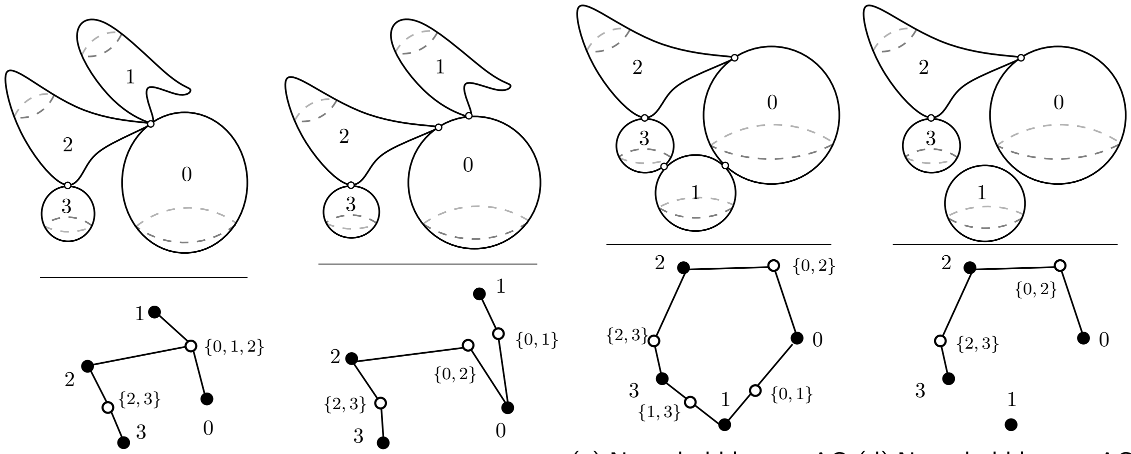


Figure III.12 – OBTS together with its adjacency graph in blue.

formally the AG of a OBTS. We say that a set A in Pow($[\![1,b]\!]$) (the powerset of the set $[\![1,b]\!]$) is a junction if

$$\bigcap_{j\in A} B_j \neq \emptyset$$



(a) A bubble topology. (b) A bubble topology. (c) Not a bubble top. AG contains a cycle. (d) Not a bubble top. AG is not connected.

Figure III.13 – At the top sketch of different topologies. At the bottom the AG graph: white points are junctions while black vertices come from each bubble.

and if A is maximal for the inclusion order; each junction point records the labels of the bubbles that intersect at the same point. We call IP the set of all junction points. At this point it is important to notice that junctions and intersection points between bubbles are in correspondence. Now we define the graph AG by specifying its vertex and edge sets.:

— The set of vertices is $IP \cup [\![1, b]\!]$

— The edge set E is the set of pairs $\{x, y\}$ where $x \in IP$, $y \in [\![1, b]\!]$ and $y \in x$.

For an idea of OBTS's see Figure III.13.

Now we consider a OBTS

$$S = B_1 \cup B_2 \cup \cdots \cup B_b. \tag{III.7}$$

as a "labeled" topological space (see Figure III.14 for an intuition), meaning that it is more than a "formal union" of bubbles since each bubble comes with an orientation and rank in a list (to be totally formal, we should have written S as a function of the b-tupple $(B_1, \ldots, B_b)$).

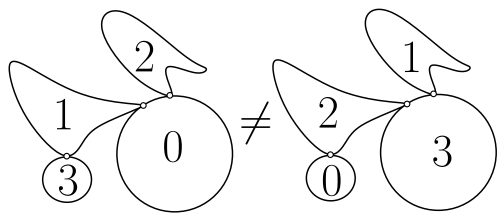


Figure III.14 – OBTS together with its adjacency graph in blue.

We say that two OBTS $S = \cup_{i=1}^{b} B_i$ and $S' = \cup_{i=1}^{b'} B_i'$ are equivalent if there exists an homeomorphism from S to S' which preserves the labelings of bubbles and the orientation of bubbles in correspondence. We call such an homeomorphism **good homeomorphism**. Notice that S equivalent to S' implies that $b = b'$ and that the AG trees are the same as labeled graphs.

We call a **pre-bubble map** a pair (D, S) where S is an OBTS and D is the drawing of a connected graph on S, such that the restriction of D to each bubble is a proper cellular drawing (as usual for maps).

Finally, we call **bubble map** the equivalence classes of pre-bubble maps for the following equivalence relation: $(D,S) \sim (D',S')$ if there exists ϕ a good homeomorphism $\phi : S \to S'$, s.t. $\phi(D) = D'$.

Returning to decorated bubble maps, let us note that the image of the path given by the walker around the tree, generates a non-self crossing circuit that passes exactly twice by each edge, more precisely, once per each oriented edge. From this circuit, it is possible to recover the original tree structure. Let us describe an algorithm that allows to recover the contour function C of the tree by using the circuit. Start at $n = 0$ and $C(0) = 0$ from the root-vertex and following the root-edge, and iterate for each new edge of the circuit:

- If the edge has been visited, set $C(n+1) = C(n) - 1$.
- If this is the first time the edge has been visited and it visits a new vertex, set $C(n+1) = C(n) + 1$.
- If this is the first time the edge has been visited and it visits an already visited vertex, set $C(n+1) = C(n) + 1$ and create a new non-self crossing circuit, where the visited vertex is duplicated. All edges visited before time n (not including the one in this step), goes with the vertex going to the right, and all the others go with the vertex going to the left. At distance $\epsilon > 0$ from this point, the graph is embedded homeomorphically as before. This can be done because the circuit is not self-crossing.
- Set $n = n + 1$.

Let us now define the image set of the gluing of a tree with a bridgeless map with a boundary. For $f, a \in \mathbb{N}^*$, we say that $(\mathfrak{m}, \mathfrak{c}) \in \mathring{\mathsf{M}}^{\mathsf{C}}_{f,a}$ if: $\mathfrak{m}$ is a circuit-decorated bubble map of f faces, $\mathfrak{c}$ is a non-crossing circuit with length $2a$, going trough each oriented edge once, passing by every pinched point of the bubble-map and containing the root edge. Let us also define $\mathring{\mathsf{M}}^{\mathsf{C},a}_{[e]}$ as $\mathring{\mathsf{M}}^{\mathsf{C}}_{f,a}$, where instead of f faces, we consider $e + a$ edges.

Let us summarize the discussion.

Proposition III.3.3.1
When one glues a tree $\mathfrak{t}' \in \mathsf{T}_a$ with a map with a (non-simple) bridgeless boundary $\mathfrak{m}^b \in \mathsf{B}_{f,2a}$, one only obtains a map only if there is no pinch point in the boundary of $\mathfrak{m}^b$ that is identified by $\mathfrak{t}'$. Moreover, g is a bijection between the Cartesian product of T_a with maps with a bridgeless boundary in $\mathsf{B}_{f,2a}$ and $\mathring{\mathsf{M}}^{\mathsf{C}}_{f,a}$.

Proof. Given the above discussion, we only need to show that we can perform the ungluing. Let us note that the tree can be recovered from the algorithm described above. To recover the map with a boundary, one just needs to duplicate the edges as described in Figure III.8. Now, we only have to explain what needs to be done close to the pinch points. In those points, one just needs to locally modify the underlying space so that pinch points give rise to circular sectors and all these circular sectors belong to the internal face. The circular sectors are determined by the circuit. □

Remark III.3.3.2
Again, the bijection does not modify the degree of interior faces, so proposition III.3.3.1 is valid when we restrict our attention to bubble q-angulations.

It can be adapted to families of maps with a given number of edges.

Proposition III.3.3.3
The function g induces a bijection (again by restriction) between:

- the set $\mathring{\mathsf{M}}^{\mathsf{C},\mathsf{a}}_{[e]}$ and
- the Cartesian product of T_a with the set of map with (non-simple) bridgeless boundary with e internal edges and boundary of size $2a$.

Let us finish this section by describing "the effect" of a boundary with bridges, which prevents the extension of g. In this case, we do not have only problems with identification of vertices, but also with the identification of edges. In particular, there may be two bridges in the boundary having oriented edges identified in the tree. This makes that the circuit we create passes at least twice by the image of that oriented edge, and thus, it makes impossible to reverse the gluing.

Additional note.

> In other words the main problem comes from the fact that one edge can be folded into itself, even more than once, giving something that is not planar nor homeomorph to a sphere.

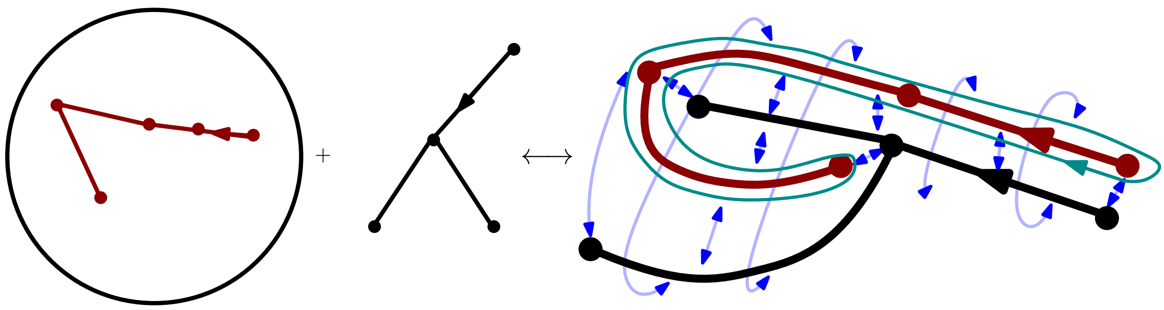


Figure III.15 – Left: Bridge map with a non-simple boundary (red tree) and a tree to glue.
Right: The gluing of the objects on the left. As in fig. III.10 the green part represents the circuit (each edge in the red tree appears twice as we follow the root edge on the boundary of the root-face). We put blue lines for identifications made by the gluing. The edges of the black tree that do not contain the root-edge generate only one edge e_1, after the gluing, what makes the green circuit visit four times e_1.

III.4 Countings

In this section, we discuss how the bijections presented before are translated into counting formulae. Before stating the results, we need to introduce a re-rooting procedure, which allows us to obtain formulas for decorated maps whose root is not necessarily in the decoration.

III.4.1 Re-rooting procedure

In this section, we explain what we call the re-rooting procedure. The motivation behind this comes from the fact that the decorated objects considered here so far have the root-edge in the decoration, while, for example in the literature, spanning-tree decorated maps have theirs root-edge in every possible oriented edge of the map. This procedure is quite general, instead we chose to explain the key idea on a particular family of objects.

Fix $r \geq 1$. A forest is a tuple of trees. We call $\mathfrak{f}$ r-**forest**, if $\mathfrak{f} = (\mathfrak{t}_i, \vec{e}_i)_{i=1}^r$, where $(\mathfrak{t}_i, \vec{e}_i)$ is a rooted tree for all $i \in \{1, 2, \dots, r\}$. We define $\mathsf{F}_{a_1,a_2,\dots,a_r}$ as the set of all r-forests such that for every $i \in \{1, 2, \dots, r\}$, $\mathfrak{t}_i$ has a_i edges.

Define a r-**boundaries map** $(\mathfrak{m}, \vec{e}_1, \vec{e}_2, \dots, \vec{e}_r)$ as a rooted map $(\mathfrak{m}, \vec{e}_1)$ and $(\vec{e}_i)_{i=1}^r$ r non intersecting edges of $\mathfrak{m}$. The map has multiple special (ordered) faces $f_1, f_2, \dots, f_r$, called the boundary faces, where f_i is the face to the left of $\vec{e}_i$; it is also required that the faces of f_i are simple and pairwise vertex-disjoint. We say that $\vec{e}_i$ is the root-edge of the i-th boundary. We denote by $\mathsf{MSB}_{f,a_1,\dots,a_r}$ the set of all maps with r-boundaries (ordered), f internal faces and with boundaries of size $a_1, \dots, a_r$.

We also define a **multiply rooted** r-**forest-decorated map** as a decorated map $(\mathfrak{m}, (\mathfrak{t}_i, \vec{e}_i)_{i=1}^r)$, where $(\mathfrak{m}, \vec{e}_1)$ is a rooted map and $(\mathfrak{t}_i, \vec{e}_i)_{i=1}^r$ is a forest with non vertex-intersecting rooted trees with $\mathfrak{t}_i \subset_M \mathfrak{m}$. Notice that from this definition the root-edge of $\mathfrak{t}_1$ coincides with the root-edge of $\mathfrak{m}$. We define $\mathring{\mathsf{M}}_f^{\mathsf{F},a_1,a_2,\dots,a_r}$ as the set of multiply rooted r-forest-decorated maps where the size of $\mathfrak{t}_i$ is a_i for all $i \in \{1, 2, \dots, r\}$.

By successively gluing the trees in the forest, we obviously have

Theorem III.4.1.1
For every $r \in \mathbb{N}^*$, there exists an explicit bijection g between: the set of multiply rooted r-forest-decorated maps $\mathring{\mathsf{M}}_f^{\mathsf{F},a_1,a_2,\dots,a_r}$ and the Cartesian product of r forests $\mathsf{F}_{a_1,a_2,\dots,a_r}$ and maps with r-boundaries $\mathsf{MSB}_{f,2a_1,2a_2,\dots,2a_r}$.

The "re-rooting procedure" is needed in order to count multiply rooted r-forest-decorated map, when instead of considering r root-edges one considers just one that could be placed in every possible oriented edge of the map. For this, let us finally define a r-**forest-decorated map** as the decorated map $((\mathfrak{m}, \vec{e}), \{\mathfrak{t}_i\}_{i \in I})$, where $(\mathfrak{m}, \vec{e})$ is a rooted map, $|I| = r$ and $\{\mathfrak{t}_i\}_{i \in I}$ is a set of non vertex-intersecting unrooted trees with $t_i \subset_M \mathfrak{m}$. Notice that we define the decorating forest $\{\mathfrak{t}_i\}_{i \in I}$ as a set. Consider an infinite vector $\vec{v} = (v_1, v_2, v_3, \dots)$ with finitely many non-zero coordinates. We define the set $\mathsf{M}_f^{\mathsf{F},\vec{v}}$ of r-forest-decorated maps, for which there are among $(\mathfrak{t}_i)_{i \in I}$, for all $j \in \mathbb{N}^*$, v_j trees with j edges.

The main result of this section is the following formula

$$|\mathring{\mathsf{M}}_f^{\mathsf{F},a_1,..,a_r}(q)| 2m(f,q) = |\mathsf{M}_f^{\mathsf{F},\vec{c}}(q)| \left(\prod_{i=1}^r 2a_i \right) \prod_k c_k!, \tag{III.8}$$

where $m(f,q)$ is the number of edges of a q-angulation with f faces, and where $\vec{c}$ is the infinite vector which encodes the multiplicity of the values a_i, i.e.

$$c_k := |\{i \in [\![1, r]\!] : a_i = k\}|, \quad k \geq 1. \tag{III.9}$$

Let us now prove Equation (III.8). It just follows from counting a slightly bigger set. Let us call

$$\tilde{\mathsf{M}}_f^{\mathsf{F},a_1,..,a_r}(q) := \left\{ (\mathfrak{m}, \vec{e}, (\mathfrak{t}_i, \vec{e}_i)_{i=1}^r) : (\mathfrak{m}, (\mathfrak{t}_i, \vec{e}_i)_{i=1}^r) \in \mathring{\mathsf{M}}_f^{\mathsf{F},a_1,..,a_r}(q) \text{ and } \vec{e} \text{ is an oriented edge of } \mathfrak{m} \right\}.$$

In other words, an element of $\tilde{\mathsf{M}}_f^{\mathsf{F},a_1,..,a_r}(q)$ is made of an element of $\mathring{\mathsf{M}}_f^{\mathsf{F},a_1,..,a_r}(q)$ together with an additional marked oriented edge $\vec{e}$ that can be oriented in any possible oriented edge (including the $\vec{e}_i$'s). We can compute the cardinality of $\tilde{\mathsf{M}}_f^{\mathsf{F},a_1,..,a_r}(q)$ in two ways. The first one is to use the definition, which gives the product between the amount of possible values of $\vec{e}$ and the cardinality of $\mathring{\mathsf{M}}_f^{\mathsf{F},a_1,..,a_r}(q)$, this gives the left-hand side of (III.8). The other one is to define a bijection between $\tilde{\mathsf{M}}_f^{\mathsf{F},a_1,..,a_r}(q)$ and $\mathsf{M}_f^{\mathsf{F},\vec{c}}(q)$ with an ordering of its trees respecting that the i-th tree has size a_i and then assigning one root edge to each tree. Therefore the cardinality of $\tilde{\mathsf{M}}_f^{\mathsf{F},a_1,..,a_r}(q)$ is obtained by the multiplication shown in the right-hand side of (III.8), where $\prod_k c_k!$ comes from the ordering needed only for the trees of each size.

Remark III.4.1.2
It is important to remark that we can do this procedure as soon as we work with two types of rootings of the same family of maps. The justification is subtle since if one unroots a given map, certain symmetries may appear, breaking down the argument. Instead, when distinguishing more than one edge, symmetries do not appear and this type of identities follow, since any automorphism of a map that fixes one oriented edge, fixes all of them..

III.4.2 Counting relation between maps with a boundary and maps with a simple boundary

The main interest of this section is to compute the generating functions of the maps with a simple boundary, as they appear in the bijection presented in Theorem III.1.2.8. To do that, we are going to adapt the technique introduced in [19] that were used to link the generating function of quadrangulations with a boundary to that of quadrangulations with a simple boundary.

Let us start by noting that the set of maps with a simple boundary and f faces is infinite. Instead, one needs to specify the number of edges and the size of the boundary.

We define the following generating functions

$$\mathcal{B}(x,y) = \sum_{e=0}^{\infty}\sum_{p=0}^{\infty} b_{e,p}x^e y^p,$$
$$\mathcal{S}(x,y) = \sum_{e=0}^{\infty}\sum_{p=0}^{\infty} s_{e,p}x^e y^p,$$

where $b_{e,p}$, resp. $s_{e,p}$, is the number of general rooted maps with e total edges and among which p edges are on the boundary, where the boundary is simple for the case of $s_{e,p}$.

Similar to [19], we obtain the following identity relying $\mathcal{S}$ and $\mathcal{B}$.

$$\mathcal{S}(x, y\mathcal{B}(x,y)) = \mathcal{B}(x,y). \tag{III.10}$$

Let us sketch the justification (see fig. III.16)): a rooted map with a general boundary $(\mathfrak{m}^b, \vec{e})$ can be decomposed for some $p \in \mathbb{N}$ into $((\mathfrak{sm}^b, \vec{e}), (\mathfrak{m}_i^b, \vec{e}_i)_{i=1}^p)$ where:

- $(\mathfrak{sm}^b, \vec{e})$ is the maximal simple boundary connected component of the root-edge which has boundary size p, for some $p \in \mathbb{N}^*$.

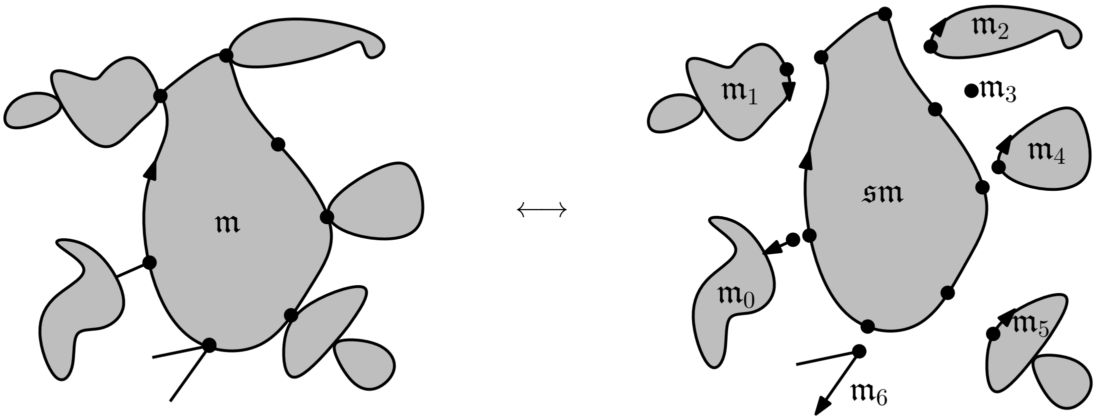


Figure III.16 – Decomposition of a rooted map with general boundary $(\mathfrak{m}, \vec{e})$ into a rooted map with simple boundary together with rooted maps with general boundary $((\mathfrak{sm}^b, \vec{e}), (\mathfrak{m}_i^b, \vec{e}_i)_{i=1}^p)$. The respective root-edges are represented by arrows.

- $(\mathfrak{m}_i^b, \vec{e}_i)$ is a rooted map with general boundary equal to the map hanging from the i-th vertex in the boundary of $\mathfrak{sm}^b$ (numbered following the root-edge) and rooted in the first edge in $\mathfrak{m}_i^b$ found when following the boundary of $\mathfrak{m}^b$. Beware this map can be reduced to a point, in such a case it is unrooted.

It turns out that this decomposition defines a bijection. Because of this, for each edge in the boundary of a general map with simple boundary we count a weight $y\mathcal{B}(x, y)$ to recover all maps with a boundary. The weight $y\mathcal{B}(x, y)$ is associated to the weight of the edge in the boundary and the map hanging from the tail of this edge consider as oriented following the contour of the simple boundary in the sense of the root-edge.

The only difference with [19], is that the here external boundary may have any possible length, while for quadrangulations the external boundary has to be of even length. When applying this technique, depending on the family of maps under study, it is important to take into account this type of restriction to obtain the right counting formulas.

Now we use Equation (III.10) to discover $\mathcal{S}$. To start with, it is well known (see, for example, [48, VII.8.2]) that $\mathcal{B}(x, y)$ satisfies

$$\mathcal{B}(x, y) = 1 + y^2 x\mathcal{B}(x, y)^2 + \frac{xy}{1-y}\left(\mathcal{B}(x, 1) - y\mathcal{B}(x, y)\right), \tag{III.11}$$

where $\mathcal{B}(x, 1)$ is the counting formula for general maps, with the following explicit form

$$\mathcal{B}(x, 1) = \sum_{e=0}^{\infty} \frac{2 \cdot 3^e}{(e+1)(e+2)} \binom{2e}{e} x^e = -\frac{1}{54x^2}\left(1 - 18x - (1-12x)^{3/2}\right). \tag{III.12}$$

Now, turning into general maps with simple boundary, make the change of variable $z = y\mathcal{B}(x, y)$ in eq. (III.11) to obtain

$$\mathcal{S}(x, z) = 1 + xz^2 + \frac{xz}{\mathcal{S}(x, z) - z}\left(\mathcal{B}(x, 1) - z\right). \tag{III.13}$$

This gives a quadratic equation for $\mathcal{S}$, obtaining the desired function $\mathcal{S}$ the only possible solution of this equation with positive coefficients

$$\mathcal{S}(x, z) = \frac{1}{2}\left(1 + z + xz^2 - \sqrt{(-xz^2 + z + 1)^2 - \frac{2z(1 + 36x - (1-12x)^{3/2})}{27x}}\right),$$

Its first coefficients are shown in the expansion $\mathcal{S}(x,z) = 1 + xz + 2x^2z + xz^2 + 9x^3z + x^2z^2 + 54x^4z + 5x^3z^2 + 378x^5z + 32x^4z + x^3z^3 + \ldots$.

This generating function encodes the number of general maps with a simple boundary, the coefficients may be recovered from it doing a sequence of derivations. Nevertheless, they do not have a closed form to our knowledge.

III.4.3 Counting results

Before presenting the results, let us start by recalling that the number of (rooted) trees with a edges is given by Catalan numbers, where C_a (see, for example, Section 1 in [11])

$$|\mathsf{T}_a| = \mathcal{C}_a := \frac{1}{a+1}\binom{2a}{a} \qquad \forall a \geq 0. \tag{III.14}$$

Tree decorated maps

Let us now obtain the formulas that directly come from the bijection for tree-decorated maps. The counting formulas obtained for maps come from [73] for the case of triangulations with multiples simple boundaries, and from [12] on the case of quadrangulations.

Corollary III.4.3.1
Set $m = (p+2a)/2$. The number of tree-decorated triangulations with boundary of size $p \geq 1$ and a tree of size a, rooted on the tree, is

$$|\mathsf{B}^{\mathsf{T},a}_{f,p}(3)| = 2^{f-2m}\frac{(3f/2+m-2)!!}{(f/2-m+1)!(f/2+3m)!!}\frac{2m}{a+1}\binom{4m}{2m}\binom{2a}{a}, \tag{III.15}$$

where $n!!$ stands from double factorial (or semifactorial). The cardinality of tree-decorated quadrangulations with boundary of size p and a tree of size a, rooted on the tree, is

$$|\mathsf{B}^{\mathsf{T},a}_{f,p}(4)| = 3^{f-m}\frac{(2f+m-1)!}{(f+2m)!(f-m+1)!}\frac{2m}{a+1}\binom{3m}{m}\binom{2a}{a} \tag{III.16}$$

Proof. This is obtained from Proposition III.3.2.1, and the comment above Lemma III.3.1.2. The formula for the number of trees with a edges and Theorem 1 of [73] and Section 2.2 of [19] respectively. □

Forest decorated maps

Corollary III.4.3.2
The cardinality of r-forest decorated triangulations, with trees of size $a_1, a_2, \ldots, a_r$, $a := \sum a_i$ and $a + r \geq f/2 + 2$ is given by

$$|\mathsf{M}^{\mathsf{F},\vec{c}}_{f}(3)| = 2^{f-2a}\frac{3f}{\prod_{k\in\mathbb{N}} c_k!}\frac{(3f/2+a-2)!!}{(f/2-a+2-r)!(f/2+3a)!!}\prod_{i=1}^{r}\frac{1}{a_i+1}\binom{4a_i}{2a_i, a_i, a_i} \tag{III.17}$$

where $\vec{c}$ is defined as in eq. (III.9).

The cardinality of r-forest decorated quadrangulations, with trees of size $a_1,\ a_2,\ldots,a_r$ and $a+r\geq f+2$, is given by:

$$|\mathsf{M}_f^{\mathsf{F},\vec{c}}(4)| = 3^{f-a}\frac{4f}{\prod_{k\in\mathbb{N}} c_k!}\frac{(2f+a-1)!}{(f+2a)!(f-a+2-r)!}\prod_{i=1}^{r}\frac{1}{a_i+1}\binom{3a_i}{a_i,a_i,a_i} \tag{III.18}$$

where $\vec{c}$ is defined as in eq. (III.9).

Proof. This is obtained from Theorem III.4.1.1, the comment above Lemma III.3.1.2, the formula for the number of trees with a_i edges and the results of [73, Theorem 1] and [12, Theorem 1.2] respectively, giving that

$$|\mathring{\mathsf{M}}_f^{\mathsf{F},a_1,..,a_r}(3)| = 2^{f-2a}\frac{(3f/2+a-2)!!}{(f/2-a+2-r)!(f/2+3a)!!}\prod_{i=1}^{r}\frac{2a_i}{a_i+1}\binom{4a_i}{2a_i,a_i,a_i} \tag{III.19}$$

and

$$|\mathring{\mathsf{M}}_f^{\mathsf{F},a_1,..,a_r}(4)| = 3^{f-a}\frac{(2f+a-1)!}{(f+2a)!(f-a+2-r)!}\prod_{i=1}^{r}\frac{2a_i}{a_i+1}\binom{3a_i}{a_i,a_i,a_i}. \tag{III.20}$$

We conclude using the re-rooting procedure condensed in (III.8). □

Remark III.4.3.3
Let us note that Corollary III.1.2.4 can be obtained from Corollary III.4.3.2 using $r=1$.

Similar counting formulas can be obtained for triangulations of girth bigger than 2 (loopless triangulations) and 3; and loopless quadrangulations (see [12]). In Corollary III.4.3.2, one could also consider a generalization for "tree-decorated maps with boundaries" an analog of the tree-decorated maps with a boundary for multiples boundaries.

As discussed in Section III.4.2, the number of general maps with a simple boundary does not have a closed formula to our knowledge, still it is possible to obtain the number of general maps decorated by a tree once extracted the coefficients $s_{e,p}$ with p even. More formally, the number of general maps decorated by a tree of size a and with e edges is given by $\mathcal{C}_a s_{e+m,2m}$, which is justified by Theorem III.1.2.8.

The cardinality of Bubble-maps can be obtained from Proposition III.3.3.3.

Corollary III.4.3.4
The cardinality of the set $\mathsf{M}_{[e]}^{\mathsf{C},a}$ of non-crossing circuit decorated bubble-maps $(\mathfrak{m},\mathfrak{c},\vec{e})$ with $e+a$ edges decorated by a circuit of size $2a$ and with root-edge $\vec{e}$ in an oriented edge of the map $\mathfrak{m}$ is

$$|\mathsf{M}_{[e]}^{\mathsf{C},a}| = 3^e\frac{(2e+2a-1)!}{e!(e+2a+1)!}\frac{2(e+a)}{a+1}\binom{4a}{2a,a,a}. \tag{III.21}$$

Proof. This is obtained from Proposition III.3.3.3, the formula for the number of trees with a edges, the results of [52, Section 2.2] together with [19, Section 2.2] and a type of re-rooting procedure, similar to the one in Section III.4.1. □

We leave to the reader the computation of the formula of maps decorated by a special type of trees, as for example trees with prescribed degree distribution, see for example [107].

Spanning tree-decorated maps

In this subsection, we discuss the consequence of our result for spanning tree-decorated maps. In this case, the counting formula of spanning tree-decorated maps was given by Mullin [93], where the root does not necessarily belong to the tree: the set of spanning tree-decorated maps with e edges are in correspondence with the set of pair of rooted trees with e and $e+1$ edges; i.e the number of spanning tree decorated maps with e edges is $C_e C_{e+1}$. Later Bernardi [114, 26, 11] gave a bijective proof of this result.

We denote by $\mathring{\mathsf{M}}_f^{\mathsf{ST}}$, resp. $\mathsf{M}_f^{\mathsf{ST}}$, the set of rooted spanning tree-decorated maps with f faces, resp. spanning tree-decorated maps with f faces. Note that in the case of triangulations we obtain that for f faces the number of vertices is $2+f/2$ and the number of edges is $3/2f$, therefore, the number of spanning-tree decorated triangulations is

$$|\mathsf{M}_f^{\mathsf{ST}}(3)| = |\mathsf{M}_f^{\mathsf{T},f/2+1}(3)| = \frac{12f}{(f+4)(f+2)^2}\binom{2f}{f, f/2, f/2}. \tag{III.22}$$

In the case of quadrangulations, the condition of having f faces implies that it has $f+2$ vertices and $2f$ edges. Therefore, we obtain the counting formula for spanning-tree decorated quadrangulations.

$$|\mathring{\mathsf{M}}_f^{\mathsf{ST}}(4)| = |\mathring{\mathsf{M}}_f^{\mathsf{T},f+1}(4)| = \frac{2}{(f+1)(f+2)}\binom{3f}{f,f,f} \tag{III.23}$$

$$|\mathsf{M}_f^{\mathsf{ST}}(4)| = |\mathsf{M}_f^{\mathsf{T},f+1}(4)| = \frac{4f}{(f+1)^2(f+2)}\binom{3f}{f,f,f}. \tag{III.24}$$

Here we recover the results from the Walsh and Lehman's Bijection [114]. In fact, the set of dual elements of $\mathring{\mathsf{M}}_f^{\mathsf{ST}}(3)$ is the set of spanning tree-decorated 3-regular maps with f vertices, and the set of dual elements of $\mathring{\mathsf{M}}_f^{\mathsf{ST}}(4)$ is the set of spanning tree-decorated 4-regular maps with f vertices. The countings appear in Section 6.2 of [18], where they make explicit the formula for more general families of maps, given that (dual) trees with prescribed degree distribution are easily counted.

Spanning r-forest decorated maps

Corollary III.4.3.5
The cardinality of spanning r-forest-decorated triangulations, with trees of size $a_1, a_2, \ldots, a_r$ and $a := \sum a_i$, is given by

$$|\mathsf{M}_f^{\mathsf{SF},\vec{c}}(3)| = 4^{r-2}\frac{3f}{\prod_{k\in\mathbb{N}} c_k!}\frac{(2f+2-r)!!}{(2f+6-3r)!!}\prod_{i=1}^{r}\frac{1}{a_i+1}\binom{4a_i}{2a_i, a_i, a_i}, \tag{III.25}$$

where $\vec{c}$ is defined as in eq. (III.9).

The cardinality of spanning r-forest-decorated quadrangulations, with trees of size $a_1, a_2, \ldots, a_r$, is given by:

$$|\mathsf{M}_f^{\mathsf{SF},\vec{c}}(4)| = 3^{r-2}\frac{4f}{\prod_{k\in\mathbb{N}} c_k!}\frac{(3f-r+1)!}{(3f-2r+4)!}\prod_{i=1}^{r}\frac{1}{a_i+1}\binom{3a_i}{a_i, a_i, a_i}, \tag{III.26}$$

where $\vec{c}$ is defined as in eq. (III.9).

Proof. A triangulation with f faces has $2+f/2$ vertices, which has to be equal to $a+r = \sum_{i=1}^{r} a_i + r$, the number of vertices covered by the forest. As before, a quadrangulation with f faces has $f+2$ vertices, which has to be equal to $a+r$. The result follows from Corollary III.4.3.2. □

Remark III.4.3.6
From this formula it is also possible to deduce Corollary III.1.2.5.

Remark III.4.3.7
Notice that we explicit the expressions for the cardinality rooted spanning-tree decorated quadrangulations. We want to point out that the right side of (III.23) looks like a possible generalization of the Catalan numbers. More rigorously, for $n, m \in \mathbb{N}$, $m \geq 1$ define:

$$\mathcal{C}_{m,n} = m! \left(\prod_{i=1}^{m} \frac{1}{n+i} \right) \binom{(m+1)n}{\underbrace{n, n, \dots, n}_{m+1 \text{ times}}} = \binom{m+n}{n}^{-1} \binom{(m+1)n}{\underbrace{n, n, \dots, n}_{m+1 \text{ times}}}.$$

When $m = 1$, we recover the Catalan numbers and, for this definition, $\mathcal{C}_{2,f}$ counts $|\mathring{\mathsf{M}}_f^{\mathsf{ST}}(4)|$. To our knowledge, this extension has not been defined so far and it does not appear in the OEIS [a].

a. $4th$ march 2019 update: it has been added for $m = 2, 3$.

From the definition it is not clear to see that $\mathcal{C}_{m,n}$ is, in fact, an integer. Luckily for us, Vincent Jugé found an analytical proof of this fact that we present in the following proposition.

Proposition III.4.3.8
For all $n,\ m \in \mathbb{N}$ and $m \geq 1$, $\mathcal{C}_{m,n}$ is integer.

Proof. Define $\nu_p(k)$ as the largest power of p prime that divides $k \in \mathbb{N}$. Recall that by Legendre's formula

$$\nu_p(k!) = \sum_{i=1}^{\infty} \left\lfloor \frac{k}{p^i} \right\rfloor.$$

Thanks to this, we can calculate the maximal power of p prime that divides $\mathcal{C}_{m,n}$

$$\begin{aligned} &\nu_p((m+1)n!) - \nu_p((n!)^{m+1}) - \nu_p((n+m)!) + \nu_p(n!) + \nu_p(m!) \\ &\qquad = \sum_{i=1}^{\infty} \underbrace{\left\lfloor \frac{(m+1)n}{p^i} \right\rfloor - (m+1) \left\lfloor \frac{n}{p^i} \right\rfloor}_{=:(1)\geq 0} \underbrace{- \left\lfloor \frac{m+n}{p^i} \right\rfloor + \left\lfloor \frac{n}{p^i} \right\rfloor + \left\lfloor \frac{m}{p^i} \right\rfloor}_{=:(2)\geq -1} \end{aligned}$$

Note that to conclude we just need to show that each term in the summation is bigger than or equal to zero. To do this notice that we just need to show that it cannot happen simultaneously $(1) = 0$ and $(2) = -1$.

Assume that this is the case and write

$$\begin{aligned} n &= kp^i + l_n, \\ m &= k'p^i + l_m, \end{aligned}$$

with $0 \leq l_n, l_m < p^i$ and $l_m < m$. Note that the fact that $(1) = 0$ implies that

$$l_n(m+1) < p^i. \tag{III.27}$$

Furthermore, the fact that $(2) = -1$ implies that $l_n \neq 0$ and

$$l_n + l_m \geq p^i.$$

Together with (III.27) this implies that $ml_n < l_m \leq m$ which implies that $l_n = 0$ so we have a contradiction. □

Another proof of this fact was given by Delphin Sénizergues as follows:

Proof.

$$\mathcal{C}_{m,n} = \left(\prod_{i=1}^{m-1} \binom{n+i}{i} \right) \times A_{n,m+1}$$

where $A_{n,m}$ counts the number of standard young tableaux of shape $\lambda = (n, n, \ldots, n)$ with m repetitions of n [56, page 29], the conclusion follows from the multiplication of natural numbers. □

IV Tree-decorated planar maps: local limits.

Contents

IV.1 Introduction

In this section we borrow the notation and definitions of section III.2 and in section III.2.2.

Define for $f, p, a \in \mathbb{N}^*$:

- $\mathsf{B}^{\mathsf{T},a}_{f,p}(q)$ the set of all (f,p,a) tree-decorated q-angulations with a simple boundary $(\mathfrak{m}^b, \mathfrak{t}, \vec{e})$, where $(\mathfrak{m}^b, \vec{e})$ is a rooted map with a simple boundary of size p having f faces with the root-edge $\vec{e}$ in the boundary and $\mathfrak{t}$ is a tree with a edges which intersects the boundary only at the root-vertex of $\mathfrak{m}^b$.
- $\mathsf{B}^{\mathsf{T},a}_{f,0}(q) := \mathsf{B}^{\mathsf{T},a}_{f}(q)$ the set of all (f,a) tree-decorated q-angulations.
- $\mathsf{SB}_{f,p}(q)$ the set of all q-angulations with a simple boundary $(\mathfrak{m}^b, \vec{e})$ where $\mathfrak{m}^b$ is a map with f faces and a simple boundary of size p.

Sometimes we will not explicitly mention the root-edge when we use objects of these sets. Set the following random variables:

$$
\begin{aligned}
Q^{\mathsf{T},a}_{f,p} &= \text{unif. r,v. in } \mathsf{B}^{\mathsf{T},a}_{f,p}(4) \text{ (quadrangulations)},\\
Q_{f,p} &= \text{unif. r,v. in } \mathsf{SB}_{f,p}(4) \text{ (quadrangulations)},\\
\tau_a &= \text{unif. r,v. in } \mathsf{T}_a,\ ^{1}\\
T^{\mathsf{T},a}_{f,p} &= \text{unif. r,v. in } \mathsf{B}^{\mathsf{T},a}_{f,p}(3) \text{ (triangulations)}\\
T_{f,p} &= \text{unif. r,v. in } \mathsf{SB}_{f,p}(3) \text{ (triangulations)}.
\end{aligned}
$$

All along this section we consider **locally finite maps (or graphs)**, which are maps (or graphs) where every vertex has finite degree. For a graph G and a subgraph $G' \subset G$, we define $G \setminus G'$ as the graph with set of vertices $V(G) \setminus V(G')$ and set of edges defined by the edges in G with both endpoints in $V(G) \setminus V(G')$.

Let G be a graph and let $(G_i)_{i\in\mathbb{N}}$ be a sequence of finite subgraphs of G, which is increasing and exhausts G, i.e. $G_i \subset G_{i+1}$ for all $i \in \mathbb{N}$ and $\cup_{i\in\mathbb{N}} G_i = G$. A decreasing sequence (C_i) of subgraphs of G is called a $(G_i)_{i\in\mathbb{N}}$-end of G if for all $i \in \mathbb{N}$, C_i is an infinite connected component of $G \setminus G_i$. It is well known that the number of $(G_i)_{i\in\mathbb{N}}$-ends does not depend on the choice of the sequence $(G_i)_{i\in G}$ (see, for example, [33]), so we just call them ends. We denote for a graph G the value Ends(G) as the number of ends in G. Ends represent "possible ways to go to infinity", more formally, ends (in the locally finite setting) can be defined as the equivalence classes of 1-way (injective) infinite paths, where two such paths are equivalent if no finite set of vertices separates them, i.e. there is no finite set of vertices such that the removal of such set separates them into two different connected components.

We will sometimes use some graph notions on maps (as ends): in this case we will naturally consider the subjacent graph of the maps under discussion.

We will study local limit of tree-decorated maps as the parameters $f, p, a \in \mathbb{N}^*$ goes to infinity, which will give infinite tree-decorated maps which will be decorated by (potentially) an infinite tree.

In our setting, 1-way infinite paths can be classified depending whether they use "effectively" or not the decoration, given that the tree decorating the map can be seen as a frontier for infinite paths.

We call D-end of a pair of graphs (G, G'), where $G' \subset G$, an end either in $G \setminus G'$ or in G'. The number of D-ends is denoted by D-ends(G, G') , i.e.

$$\text{D-ends}(G, G') = \text{Ends}(G \setminus G') + \text{Ends}(G').$$

Since the random variable $Q^{\mathsf{T},a}_{f,p}$ consists of a random pair of maps, where the second map is a subgraph of the first one, it makes sense to write D-ends$(Q^{\mathsf{T},a}_{f,p})$.

Our motivation to introduce this new notion of "end" comes from the fact that depending on the type of decoration the local limits that we will obtain have different D-ends values.

Local topology

For a map $\mathfrak{m}$ and $r \in \{0, 1, 2, \dots\}$, let $[\mathfrak{m}]_r$ denote the map obtained by considering all the faces of $\mathfrak{m}$ whose vertices are all at graph distance smaller than r from the root-vertex in $\mathfrak{m}$.

Let $\mathcal{S}$ be a family of finite maps. The **local topology** on $\mathcal{S}$ is the topology induced by the metric d_{loc}, where

$$\mathsf{d}_{loc}(\mathfrak{m}_1, \mathfrak{m}_2) = (1 + \sup\{r \geq 0 : [\mathfrak{m}_1]_r = [\mathfrak{m}_2]_r\})^{-1}$$

i.e. the distance is equal to the inverse of the first ball radius where both maps disagree. It is not difficult to prove that the space $(\overline{\mathcal{S}}, \mathsf{d}_{loc})$, where $\overline{\mathcal{S}}$ denotes the completion of $\mathcal{S}$ with respect to d_{loc}, is Polish (metric, separable and complete). By definition of the distance, $(\mathfrak{m}_n)$ converges if for all $r \in \{0, 1, 2, 3, \dots\}$, $[\mathfrak{m}_n]_r$ is a constant sequence from a certain point on.

1. The set of rooted planar trees with a edges.

This topology ensures that limit of locally finite graphs is locally finite too, since sequences where the degree of a vertex grows indefinitely do not have constant balls from a certain point on. The completion is notably used since $\mathcal{S}$ may possibly be a family of finite maps with sequences converging to infinite maps for this metric. We specify with $\xrightarrow{(local)}$ the convergence for the local topology and $\xrightarrow{(d)}$ for convergence in distribution.

One of the first results concerning the convergence for the local topology of random combinatorial objects is due to Kesten [70] and says for τ_n defined in the intro

$$\tau_n \xrightarrow[local]{(d)} \tau_\infty \tag{IV.1}$$

where τ_∞ is the **critical geometric Galton-Watson tree conditioned to survive**. The random variable τ_∞ is an infinite rooted plane tree with a.s. one infinite branch called the spine (i.e. it is one-ended). It can be constructed as follows: Consider a sequence of i.i.d. random variables $(G_i)_{i\in\mathbb{N}^*}$ distributed according to a size-biased geometric distribution with parameter $1/2$ for all $i \in \{1,2,3,\dots\}$, so that $\mathbb{P}(G_i = k) = k2^{-k}$, $k \geq 0$; and set $G_0 = 1$. Define the *preliminary i-th generation* as an ordered set of nodes with size G_i. The tree τ_∞ is obtained by:

- linking each element in the i-th preliminary generation with I_{i-1} (in an ordered an non-intersecting fashion) and
- graft at each element in the i-th preliminary generation, but I_i, independent $1/2$-geometric Galton-Watson trees, which are a.s. finite since they are critical Galton-Watson trees.

(for a detailed explanation, see [70]).

In the setting of random triangulations with a simple boundary, Angel [5] obtained that

$$T_{f,p} \xrightarrow[local\ (f\to\infty)]{(d)} T_{\infty,p} \xrightarrow[local\ (p\to\infty)]{(d)} \mathcal{H}(3)_\infty, \tag{IV.2}$$

where $\mathcal{H}(3)_\infty$ is the **Uniform infinite half plane triangulation with simple boundary** (also denoted as the UIHPT). Both $T_{\infty,p}$ and $\mathcal{H}(3)_\infty$ are one-ended. For $p = 2$ the limiting object $T_{\infty,1}$ is called **Uniform infinite plane triangulation** UIPT.

In the setting of random quadrangulations with a simple boundary, Curien & Miermont [30] proved that

$$Q_{f,p} \xrightarrow[local\ (f\to\infty)]{(d)} Q_{\infty,p} \xrightarrow[local\ (p\to\infty)]{(d)} \mathcal{H}(4)_\infty, \tag{IV.3}$$

where $\mathcal{H}(4)_\infty$ is the **Uniform infinite half plane quadrangulation with simple boundary** (also denoted as the UIHPQ$_S$). As in the case of triangulations, $Q_{\infty,p}$ and $\mathcal{H}(4)_\infty$ are one-ended. For $p = 2$ the first convergence is due to Krikun [72] and the limiting object $Q_{\infty,1}$ is the well known **Uniform infinite plane quadrangulation** UIPQ.

Another local limit result, important in the sequel, is due to Caraceni & Curien [21]. They stated that $Q^{(k)}_{\infty,p}$ defined as the re-rooting at the k-th edge along the boundary of the external face of $Q_{\infty,p}$ is asymptotically independent from $Q_{\infty,p}$ as p and k tends to infinity in a coherent way, formally

$$(Q_{\infty,p}, Q^{(k)}_{\infty,p}) \xrightarrow[local]{(d)} (\mathcal{H}(4)_\infty, \mathcal{H}(4)'_\infty) \qquad \text{as } k\to\infty \text{ and } 2p-k\to\infty, \tag{IV.4}$$

with $\mathcal{H}(4)_\infty$ and $\mathcal{H}(4)'_\infty$ are two independent UIHPQ$_S$. This property follows from invariance under re-rooting, the one-ended behavior and the spatial Markov property (see [21, Lemma 6]); since these properties are also valid in the case of triangulations, the same proof applies in this context.

Theorem IV.4 can also be extended for multiple re-rooting as follows: for every $r \in \mathbb{N}$ and $k_0 := 0 < k_1 < k_2 < \cdots < k_r < k_{r+1} := 2p$, with $k_{i+1} - k_i \to \infty$ for all $i \in \{1, 2, \ldots, r\}$, we have

$$(Q_{\infty,p}, Q_{\infty,p}^{(k_1)}, \ldots, Q_{\infty,p}^{(k_r)}) \xrightarrow[local]{(d)} (\mathcal{H}(4)_\infty, \mathcal{H}(4)_\infty^1, \ldots, \mathcal{H}(4)_\infty^r) \tag{IV.5}$$

where $(\mathcal{H}(4)_\infty, \mathcal{H}(4)_\infty^1, \ldots, \mathcal{H}(4)_\infty^r)$ are $r+1$ independent copies of UIHPQ$_S$.

We will use sometimes the **product of local topologies** $(\mathcal{S}_1 \times \mathcal{S}_2, \mathsf{d}_{prod})$ of two local topologies $(\mathcal{S}_1, \mathsf{d}_{loc})$ and $(\mathcal{S}_2, \mathsf{d}_{loc})$ with d_{prod} defined as

$$\mathsf{d}_{prod}((\mathfrak{m}_1, \mathfrak{m}_2), (\mathfrak{m}'_1, \mathfrak{m}'_2)) = \max\{\mathsf{d}_{loc}(\mathfrak{m}_1, \mathfrak{m}'_1), \mathsf{d}_{loc}(\mathfrak{m}_2, \mathfrak{m}'_2)\} \qquad \forall \mathfrak{m}_1, \mathfrak{m}'_1 \in \mathcal{S}_1, \forall \mathfrak{m}_2, \mathfrak{m}'_2 \in \mathcal{S}_2.$$

Naturally this distance can be defined for product topologies of more than 2 metric spaces, we will use the same notation by writing d_{prod} in these cases too.

Here we work with decorated objects meaning that we consider pairs of maps $(\mathfrak{m}_1, \mathfrak{m}_2)$ such that $\mathfrak{m}_2 \subset_M \mathfrak{m}_1$. Because of this we need to make sense of a decorated-maps local topology, for this purpose we follow [25]. For a decorated map $(\mathfrak{m}_1, \mathfrak{m}_2)$ and $r \in \{0, 1, 2, \ldots\}$ we define $[\mathfrak{m}_1, \mathfrak{m}_2]_r$ as the decorated map consisting in all vertices at distance at most r from the root-vertex in $\mathfrak{m}_1$ and edges in $\mathfrak{m}_1$ between those vertices. The edges belonging to the decoration of $[\mathfrak{m}_1, \mathfrak{m}_2]_r$ are the edges of $\mathfrak{m}_2$ with endpoints at distance at most r from the root-edge of $\mathfrak{m}_1$. Let $\mathcal{DS}$ be a family of decorated maps. The **decorated-maps local topology** on $\mathcal{DS}$ is the metric space $(\mathcal{DS}, \mathsf{d}_{dec})$, where

$$\mathsf{d}_{dec}((\mathfrak{m}_1, \mathfrak{m}_2), (\mathfrak{m}'_1, \mathfrak{m}'_2)) = \inf\{2^{-r} : r \in \mathbb{N},\ [\mathfrak{m}_1, \mathfrak{m}_2]_r = [\mathfrak{m}'_1, \mathfrak{m}'_2]_r\}$$

Again we consider the polish space $(\overline{\mathcal{DS}}, \mathsf{d}_{dec})$.

IV.2 Results

We will prove the following two propositions

Proposition IV.2.0.1
For $p_n \to p \in \mathbb{N}^+ \cup \{\infty\}$ and $a_n \to a \in \mathbb{N}^+$, we have

$$T_{f,p_n}^{\mathsf{T},a_n} \xrightarrow[local\ (f\to\infty)]{(d)} T_{\infty,p_n}^{\mathsf{T},a_n} \xrightarrow[local\ (n\to\infty)]{(d)} T_{\infty,p}^{\mathsf{T},a}.$$

All these limit objects are random tree-decorated infinite triangulations with a simple boundary. In this cases

$$\text{D-ends}(T_{\infty,p}^{\mathsf{T},a}) = 1.$$

The result is also valid for quadrangulations when changing all T's by Q's in the statement.

Proposition IV.2.0.2
For $p_n \to p \in \mathbb{N}^+ \cup \{\infty\}$ and $a_n \to a \in \mathbb{N}^+ \cup \{\infty\}$

$$T^{\mathsf{T},a_n}_{\infty,p_n} \xrightarrow[local]{(d)} T^{\mathsf{T},a}_{\infty,p}.$$

All these limits are infinite triangulations, but depending on p and a

$$\text{D-ends}(T^{\mathsf{T},a}_{\infty,p}) = \begin{cases} 1 & \text{if } p < \infty, a < \infty \\ 2 & \text{if } p < \infty, a = \infty \\ 3 & \text{if } p = \infty, a = \infty \end{cases}.$$

The result is also valid for quadrangulations.

The scheme that we will follow to prove Propositions IV.2.0.1 and IV.2.0.2 is a usual technique to prove convergence in distribution for the local topology when the sequence μ_n of distributions on a set of combinatorial objects $B_n \subset B$ is the pushforward measure, by a bijection $f : A \to B$, of a measure ν_n on some set of encoding objects $A_n \subset A$. The ingredients that we need to ensure this convergence are:

1. The convergence in distribution for the topology on A of the sequence $(\nu_n)_{n\in\mathbb{N}}$.
2. The continuity of the bijection f.
3. Apply the continuous mapping theorem, which says: under continuous functions with negligible set of discontinuities for the limiting measure, the image of any converging sequence converges.

Actually if n denotes the size of the objects into play, when $n \to \infty$, naturally the limiting object will have infinite size. We will find the support of the limiting measure μ of the sequence μ_n, which is a set of infinite objects, by taking the limit of ν_n and by extending continuously the function f to infinite objects.

So our strategy will rely on a "extension by continuity" of the gluing function g on the closure of the set of finite maps where g was originally defined.

We start by recalling the gluing procedure g defined in Section III.3.2: Given $((\bar{\mathfrak{m}}^b, \vec{e}_b), (\bar{\mathfrak{t}}, \vec{e}_t))$, where $(\bar{\mathfrak{m}}^b, \vec{e}_b)$ is a rooted map with a simple boundary of length $p + 2a$ and $(\bar{\mathfrak{t}}, \vec{e}_t)$ is a tree with a edges, the map $g((\bar{\mathfrak{m}}^b, \vec{e}_b), (\bar{\mathfrak{t}}, \vec{e}_t)) = (\mathfrak{m}^b, \mathfrak{t}, \vec{e})$ a tree-decorated rooted map with a boundary of size p (possibly with $p = 0$) obtained by identifying the first $2a$ edges of the external face, starting from the root edge of $\bar{\mathfrak{m}}^b$, with the $2a$ oriented edges of the tree. The identification is done one by one following the contour of the tree $\bar{\mathfrak{t}}$ from the right of the root-edge (see fig. IV.1). The root-edge $\vec{e}$ is the image of the edge $2a + 1$ of $\bar{\mathfrak{m}}^b$ when $p > 0$ and the root of the original tree when $p = 0$.

IV.2.1 Extension of the gluing for tree-decorated infinite maps decorated in finite trees

We will extend the gluing procedure g, to be defined on a set with some infinite objects. Consider the following sets:

$$\mathsf{D} = \bigcup_{\substack{f,a\in\mathbb{N}^* \\ p\in\mathbb{N}}} (\mathsf{SB}_{f,p+2a} \times \mathsf{T}_a),$$

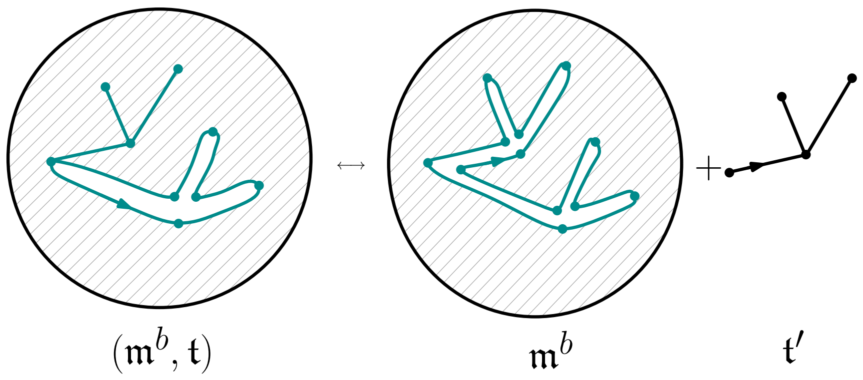


Figure IV.1 – Sketch of the correspondence in proposition III.3.2.1.

$$\mathsf{E} = \bigcup_{\substack{a\in\mathbb{N}^*\\ p\in\mathbb{N}\cup\{\infty\}}} (\mathsf{SB}_{\infty,p+2a} \times \mathsf{T}_a),$$
$$\mathsf{TB} = \bigcup_{\substack{f\in\mathbb{N}^*\cup\{\infty\},a\in\mathbb{N}^*\\ p\in\mathbb{N}\cup\{\infty\}}} \mathsf{B}^{\mathsf{T},p}_{f,a}.$$

The set D is the set of finite maps where the gluing function g is already defined. The set E consists of maps with infinite number of faces and potentially infinite boundary. Finally, the set TB is the set containing the possible images, i.e. tree-decorated maps with simple boundary with possibly infinite number of faces and infinite perimeter.

We now extend $g : \mathsf{D} \cup \mathsf{E} \to \mathsf{TB}$ as follows:

- For elements in D, same definition as before.

- For element in E, we define the function as the gluing of the first $2a$ edges in the boundary following the root edge, to the tree following its contour, obtaining in this way a tree-decorated infinite map.

Lemma IV.2.1.1
The function g is continuous from $(\mathsf{D} \cup \mathsf{E}, \mathsf{d}_{prod})$ to $(\mathsf{TB}, \mathsf{d}_{dec})$.

Proof. This is simply a consequence that the function acts on finitely many edges surrounding the root-edge and the fact that for the local topology, balls are eventually constant. □

Proof of Proposition IV.2.0.1. It follows from eq. (IV.2), the convergence of τ_{a_n} to τ_a and lemma IV.2.1.1. □

IV.2.2 New extension of the gluing for tree-decorated maps with a one-ended tree

Now we explore the gluing of a map with simple infinite boundary and a one-ended tree.

Notice that one-ended trees cannot be encoded by just one contour function, since a contour function never crosses the spine, but they can be encoded by two contour functions, the left and right contour functions, obtained using walks around the tree following the left (resp. right) side of the root-edge. These walks are represented in the middle of fig. IV.2.

For $k, \bar{k} \in \mathbb{N}$ with $k < \bar{k}$ and a given map $\mathfrak{m}^b$ with a simple boundary of size $\bar{k}$, we denote by $\mathfrak{m}^{b,(k)}$ the re-rooting of $\mathfrak{m}^b$ at the k-th edge, following the counter-sense of the root-edge, along the boundary of the root face.

Consider the following sets:

$$\begin{aligned}
\mathsf{D}_0 &= \{(\mathfrak{m}^b, \mathfrak{m}^{b,(p)}, \mathfrak{t}, p) : (\mathfrak{m}^b, \mathfrak{t}) \in \mathsf{D}, p \in \mathbb{N}\},\\
\mathsf{E}_0 &= \{(\mathfrak{m}^b, \mathfrak{m}^{b,(p)}, \mathfrak{t}, p) : (\mathfrak{m}^b, \mathfrak{t}) \in \mathsf{SB}_{\infty,p+2a} \times \mathsf{T}_a, \text{ for } a \in \mathbb{N}^*, p \in \mathbb{N}\},\\
\mathsf{E}_1 &= \{(\mathfrak{m}^b, \mathfrak{m}^{b,(p)}, \mathfrak{t}, p) : (\mathfrak{m}^b, \mathfrak{t}) \in \mathsf{SB}_{\infty,\infty} \times \mathsf{T}_\infty, p \in \mathbb{N}\},\\
\mathsf{E}_\infty &= \{(\mathfrak{m}^b, \mathfrak{m}^{b'}, \mathfrak{t}, \infty) \in \mathsf{SB}_{\infty,\infty} \times \mathsf{SB}_{\infty,\infty} \times \mathsf{T}_\infty\},\\
\mathsf{TB}_0 &= \bigcup_{\substack{f,a \in \mathbb{N}^* \cup \{\infty\}\\ p \in \mathbb{N} \cup \{\infty\}}} \mathsf{B}^{\mathsf{T},p}_{f,a}.
\end{aligned}$$

We extend the function g using g_1, which takes four arguments instead of pairs. More precisely, $g_1 : \mathsf{D}_0 \cup \mathsf{E}_0 \cup \mathsf{E}_1 \cup \mathsf{E}_\infty \to \mathsf{TB}_0$, which is defined as follows:

• For elements in D_0 and E_0 we define

$$g_1(\mathfrak{m}^b, \mathfrak{m}^{b,(p)}, \mathfrak{t}, p) = g(\mathfrak{m}^b, \mathfrak{t}).$$

• Define the function g_1 in E_1 as follows: consider $(\bar{\mathfrak{m}}_{\infty,\infty}, \bar{\mathfrak{m}}^{(p)}_{\infty,\infty}, \bar{\mathfrak{t}}, p) \in \mathsf{E}_1$, where $\bar{\mathfrak{m}}_{\infty,\infty}$ is one infinite rooted map with infinite simple boundary and $\bar{\mathfrak{t}}$ is a one-ended rooted tree. Here instead of considering $\bar{\mathfrak{m}}^{(p)}_{\infty,\infty}$ as another copy of $\bar{\mathfrak{m}}_{\infty,\infty}$, we use it to denote exactly the same map seen from another root, i.e. the correct way to read the arguments in the function is $(\bar{\mathfrak{m}}_{\infty,\infty}, \bar{\mathfrak{t}}, p)$. We define $g_1(\bar{\mathfrak{m}}_{\infty,\infty}, \bar{\mathfrak{m}}^{(p)}_{\infty,\infty}, \bar{\mathfrak{t}}, p)$ as the following decorated map:

1. we glue the right side boundary of $\bar{\mathfrak{m}}^{(p)}_{\infty,\infty}$, starting from its root-vertex, to the left of the tree $\bar{\mathfrak{t}}$ following the left contour; and
2. we glue the left side boundary of [the same map but starting from another root] $\bar{\mathfrak{m}}_{\infty,\infty}$, starting from its root-vertex and following its root-edge, to the right of the tree $\bar{\mathfrak{t}}$ following the right contour.
3. The root of the resulting map is defined to be the image of the root-edge of $\bar{\mathfrak{m}}^{(p)}_{\infty,\infty}$ under the gluing.

This construction produces a map with an external boundary of size p (coming from the external boundary between the two rootings). see fig. IV.2 for an sketch of this extension.

• Define the function g_1 in E_∞ as follows: consider $(\bar{\mathfrak{m}}_{\infty,\infty}, \bar{\mathfrak{m}}'_{\infty,\infty}, \bar{\mathfrak{t}}, \infty)$, where $\bar{\mathfrak{m}}_{\infty,\infty}$ and $\bar{\mathfrak{m}}'_{\infty,\infty}$ are two infinite rooted maps with infinite simple boundary and where $\bar{\mathfrak{t}}$ is a one ended rooted tree. We define $g_1(\bar{\mathfrak{m}}_{\infty,\infty}, \bar{\mathfrak{m}}'_{\infty,\infty}, \bar{\mathfrak{t}}, p)$ as the following decorated map:

1. we glue the right side boundary of $\bar{\mathfrak{m}}_{\infty,\infty}$, starting from its root-vertex, to the left of the tree following the left contour; and
2. we glue the left side boundary of $\bar{\mathfrak{m}}'_{\infty,\infty}$, starting from its root-vertex and following its root-edge, to the right of the tree following the right contour.
3. The root of the resulting map is defined to be the image of the root-edge of $\bar{\mathfrak{m}}_{\infty,\infty}$ under the gluing.

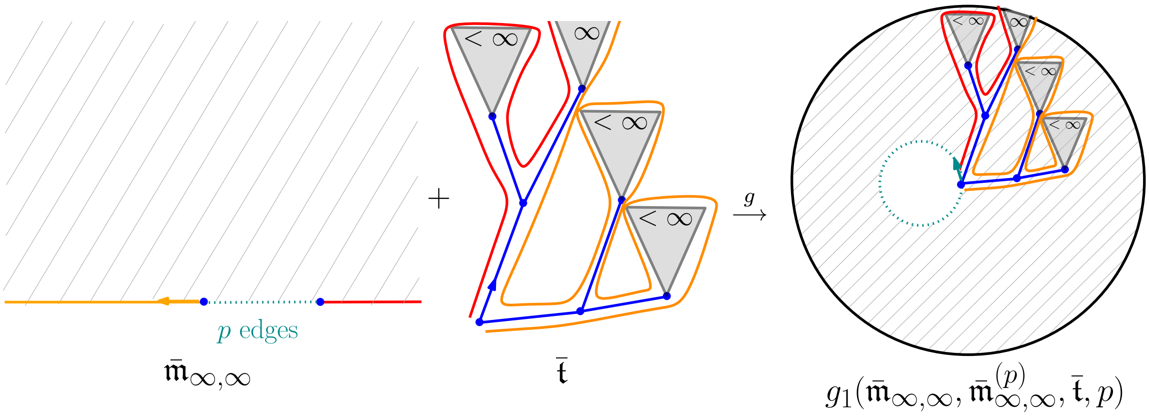


Figure IV.2 – Sketch of the definition of g_1 in E_1.

The image is an infinite tree-decorated map with an infinite simple boundary, where the two initial maps are at the left and right of the tree, "half" of their boundaries have been glued to the tree and "half" remain. See fig. IV.3 for an sketch of this extension.

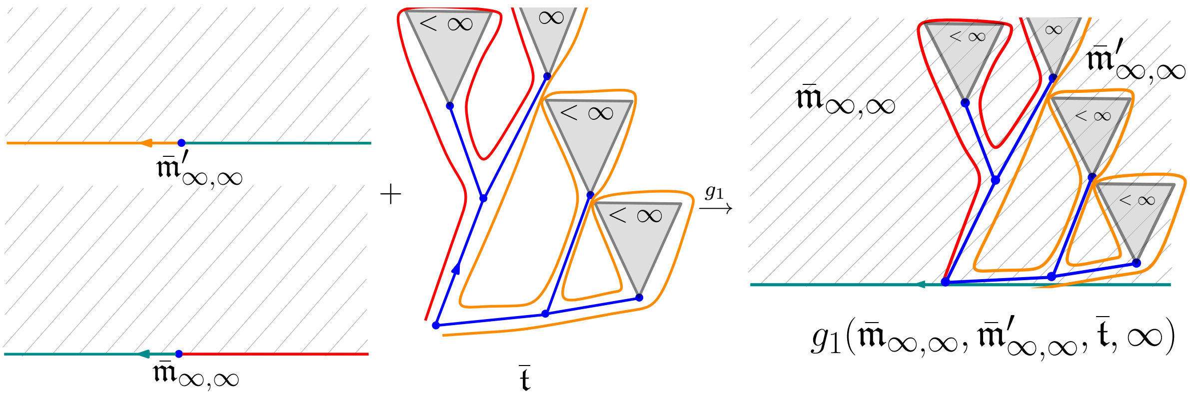


Figure IV.3 – Sketch of the definition of g_1 in E_∞. .

Consider the following distance on $\mathbb{N} \cup \{\infty\}$:

$$\mathsf{d}^*(p, p') = \left| \frac{1}{1+p} - \frac{1}{1+p'} \right|$$

It can be checked that: for any sequence $(p_n)_{n\in\mathbb{N}}$ in $(\mathbb{N} \cup \{\infty\}, \mathsf{d}^*)$

1. If $p_n \to p \in \mathbb{N}$, then the sequence is eventually constant.
2. If $p_n \to \infty$ for d^*, then $p_n \to \infty$ in the usual sense.

We define d^*_{prod} a metric on $\mathsf{D}_0 \cup \mathsf{E}_0 \cup \mathsf{E}_1 \cup \mathsf{E}_\infty$ as follows:

$$\mathsf{d}^*_{prod}((\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{t}_1, p_1), (\mathfrak{m}'_1, \mathfrak{m}'_2, \mathfrak{t}'_1, p'_1)) = \max\{\mathsf{d}_{loc}(\mathfrak{m}_1, \mathfrak{m}'_1), \mathsf{d}_{loc}(\mathfrak{m}_2, \mathfrak{m}'_2), \mathsf{d}_{loc}(\mathfrak{t}, \mathfrak{t}'), \mathsf{d}^*(p, p')\}.$$

Lemma IV.2.2.1
The function g_1 is continuous from $(\mathsf{D}_0 \cup \mathsf{E}_0 \cup \mathsf{E}_1 \cup \mathsf{E}_\infty, \mathsf{d}^*_{prod})$ to $(\mathsf{TB}, \mathsf{d}_{dec})$.

Proof of Lemma IV.2.2.1. Let x be an element of $\mathsf{D}_0 \cup \mathsf{E}_0 \cup \mathsf{E}_1 \cup \mathsf{E}_\infty$:

- It is easy to prove that x is a point of continuity if x belongs to the discrete sets D_0 or E_0.
- We will now concentrate in $x = (\bar{\mathfrak{m}}^b, \bar{\mathfrak{m}}^{b'}, \bar{\mathfrak{t}}, p)$ belonging to E_1 and E_∞. We will show that if

$$\bar{\mathfrak{m}}_n^b \xrightarrow[local]{} \bar{\mathfrak{m}}^b \quad , \quad \bar{\mathfrak{m}}_n^{b,(p_n)} \xrightarrow[local]{} \bar{\mathfrak{m}}^{b'} \quad , \quad \bar{\mathfrak{t}}_n \xrightarrow[local]{} \bar{\mathfrak{t}} \quad \text{and} \quad p_n \xrightarrow[\mathsf{d}^*]{} p \quad \text{as} \quad n \to \infty, \tag{IV.6}$$

then,

$$g_1(\bar{\mathfrak{m}}_n^b, \bar{\mathfrak{m}}_n^{b,(p_n)}, \bar{\mathfrak{t}}_n, p_n) \xrightarrow[(\mathsf{d}_{dec})]{} g_1(\bar{\mathfrak{m}}^b, \bar{\mathfrak{m}}^{b'}, \bar{\mathfrak{t}}, p) \quad \text{as } n \to \infty.$$

For this purpose it suffices to prove that for all fixed $r \in \mathbb{N}$, the balls $[g_1(\bar{\mathfrak{m}}_n^b, \bar{\mathfrak{m}}_n^{b,(p_n)}, \bar{\mathfrak{t}}_n, p_n)]_r$ and $[g_1(\bar{\mathfrak{m}}^b, \bar{\mathfrak{m}}^{b'}, \bar{\mathfrak{t}}, p)]_r$ coincide for all n large enough.

From (IV.6) follows that for every $r \in \mathbb{N}$ there exists $N_r \in \mathbb{N}$ such that for all $n \geq N_r$

$$[\bar{\mathfrak{m}}_n^b]_r = [\bar{\mathfrak{m}}^b]_r \quad , \quad [\bar{\mathfrak{m}}_n^{b,(p_n)}]_r = [\bar{\mathfrak{m}}^{b'}]_r \quad \text{and} \quad [\bar{\mathfrak{t}}_n]_r = [\bar{\mathfrak{t}}]_r.$$

There are two possible cases:

- $p \in \mathbb{N}$ and therefore, $\mathfrak{m}^{b'} = \mathfrak{m}^{b,(p)}$ and $x \in \mathsf{E}_1$.
- $p = \infty$ and therefore, $x \in \mathsf{E}_\infty$.

The following argument applies to both cases:
Fix $r \in \mathbb{N}$ and notice that since $\bar{\mathfrak{t}}$ is one ended, every vertex has finite right or finite left rank (apparition index in the right and left contour). We say that a vertex in the tree is a right (resp. left) vertex of the tree if its right (resp. left) rank is finite.

Consider r_L as the length of the left contour interval that covers all the left vertices of $\bar{\mathfrak{t}}$ in the ball of radius r; and define r_R for the right. Since the tree $\bar{\mathfrak{t}}$ is locally finite, both r_L and r_R are finite. Define $\bar{r} = r_L + r_R$. From the definition of left and right contours and of the gluing procedure $[g_1(\bar{\mathfrak{m}}_n^b, \bar{\mathfrak{m}}_n^{b,(p_n)}, \bar{\mathfrak{t}}_n, p_n)]_r$ is completely determined from $[\bar{\mathfrak{m}}_n^b]_{\bar{r}}$, $[\bar{\mathfrak{m}}_n^{(p_n)}]_{\bar{r}}$ and $[\bar{\mathfrak{t}}_n]_{\bar{r}}$, but these balls remain constant for all $n \geq N_{\bar{r}}$, proving the assertion. □

Proof of Proposition IV.2.0.2. The proof follows eqs. (IV.1), (IV.2) and (IV.4) and Lemma IV.2.2.1.□

IV.2.3 Extension for tree-decorated maps to the case of trees with multiple ends.

Up to this moment we handled one ended decorations. A natural question that arises is: can we consider more than one-ended trees as decoration? The answer to this question is yes.

We give an algorithm to partition $\mathfrak{t}$, a i-ended tree, into i one-ended trees $(\mathfrak{t}_j)_{j=1}^r$. Notice that i ended trees possesses i infinite injective paths starting from the root, such that two different paths have finite intersection.

As in usual algorithm descriptions, we will allow reassignment of variables.

1. Label the paths from left to right starting from the root-edge. It may be observed that the i-th meets the $i-1$-th path.
2. From $j = i$ to $j = 2$ (one by one).

(a) Call $\mathfrak{t}_j$ the one-ended subtree associated to the j-th path from the last point of intersection x_j with the $j-1$-th path ($\mathfrak{t}_j$ contains the portion of the infinite path from x_j that contains all the nodes that have an ancestor on this path).

(b) Record c_j as the corner where the subtree $\mathfrak{t}_j$ is attached on $\mathfrak{t}$.

(c) Set $\mathfrak{t} := \mathfrak{t} \setminus \mathfrak{t}_j$, the new tree $\mathfrak{t}$ is a $j-1$ ended tree.

3. The remaining tree $\mathfrak{t}$ after this loop has one end: it is called $\mathfrak{t}_1$.

Every tree $\mathfrak{t}_j$ for $j \in \{1, \ldots, i-1\}$ is marked at a corner c_{j+1} (where was attached $\mathfrak{t}_{j+1}$). See fig. IV.4 for a sketch of this partitioning.

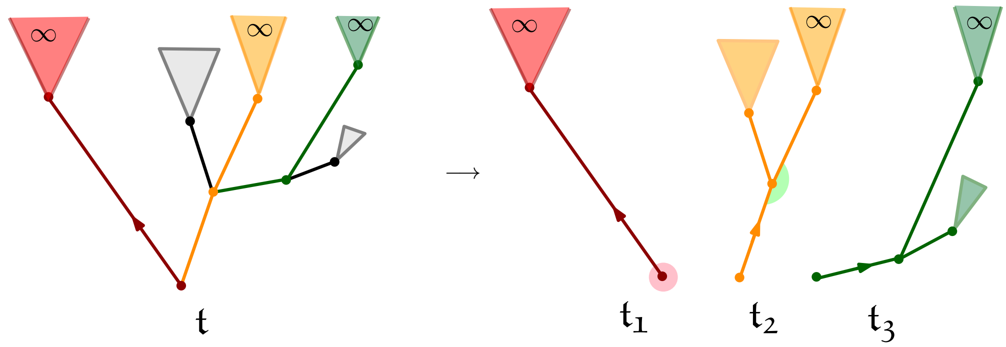


Figure IV.4 – Sketch of the the tree decomposition applied to a tree with 3-ends. The recorded corner are represented by circular sectors.

Gluing of $i+1$ infinite maps with infinite simple boundaries with an i ended tree.

We define the gluing function g_i which takes as argument $i+1$ infinite rooted maps $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_{i+1}$ with infinite simple boundary and a i ended rooted tree $\mathfrak{t}_i$ and takes values in the set of tree-decorated maps decorated in a tree with i ends.

Consider the sequence c_j of corners and trees $\mathfrak{t}_j$ obtained in the preceding decomposition and do as follows:

1. Glue $\mathfrak{m}_1$ from the right of the root-edge to the left of the tree $\mathfrak{t}_1$.
2. From $j = 1$ to $j = i-1$
   (a) Glue $\mathfrak{m}_{j+1}$ from the left of the root-edge to the right of the tree $\mathfrak{t}_j$ up to the corner c_{j+1}.
   (b) Glue $\mathfrak{m}_{j+1}$ from the right of the root-edge to the left of the tree $\mathfrak{t}_{j+1}$.
3. Glue $\mathfrak{m}_{i+1}$ from the left of the root-edge of to the right of the tree $\mathfrak{t}$ starting from the root-vertex.

The set of faces of the resulting map is the union of the sets of internal faces of $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_{i+1}$. The result has infinite boundary and is decorated by $\mathfrak{t}$. This construction is reversible.

This can also be adapted for i-ended trees and $p < \infty$.

Proposition IV.2.3.1
Consider a family of random rooted trees τ_n^i such that

$$\tau_a^i \xrightarrow[local]{(d)} \tau_\infty^i \quad \text{as } a \to \infty,$$

where τ_∞^i has a.s. i ends.

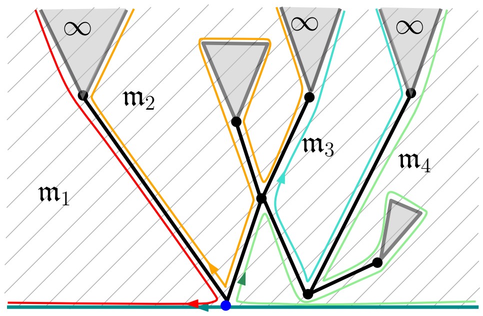


Figure IV.5 – Right: sketch of the gluing for $i = 3$. The boundaries of $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{m}_3$, $\mathfrak{m}_4$ is sketched in red, orange, light blue and light green respectively. The respective root edges are drawn as arrows.

Then, for $p_n \to p \in \mathbb{N}^+ \cup \{\infty\}$ and $a_n \to \infty$, the gluing by g_i between T_{∞, p_n+2a_n} and $\tau^i_{a_n}$ converges for the local topology and the limit is described by g_i of $i+1$ independent UIHPTand τ^i_∞. An analog result is also valid for quadrangulations.

Remark IV.2.3.2
To get this result a suitable notion of topology is needed to take into account all the elements as $(\mathfrak{m}_j)_{j=1}^{i+1}$, $(\mathfrak{t}_j)_{j=1}^{i}$, $\{c_j\}_{j=2}^{i}$, $(p_n)_{n\in\mathbb{N}}$. This is an adaptation of the proof of continuity of g_1 (here we use that c_i converges given that in the limit the trees are i ended and two different infinite paths intersect in finitely many elements).

A representations of the local limit in the proposition IV.2.3.1 is presented in fig. IV.5.

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