Bijections for tree-decorated map and applications to random maps.

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Work with J. Bettinelli, N. Curien and A. Sepúlveda

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Tree-decorated planar maps: combinatorial results.

with A. Sepúlveda.

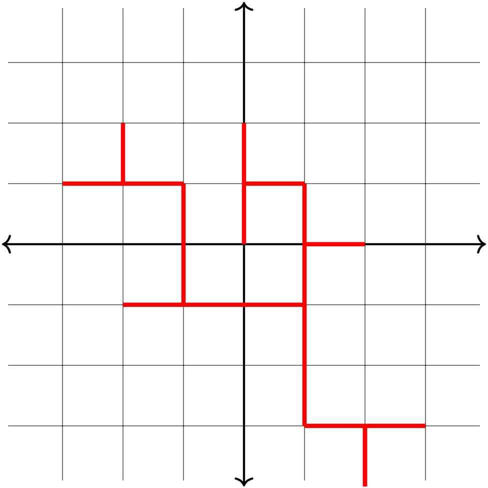

Figure: Uniform random tree of size 20 containing the origin on $\mathbb{Z}^2$.

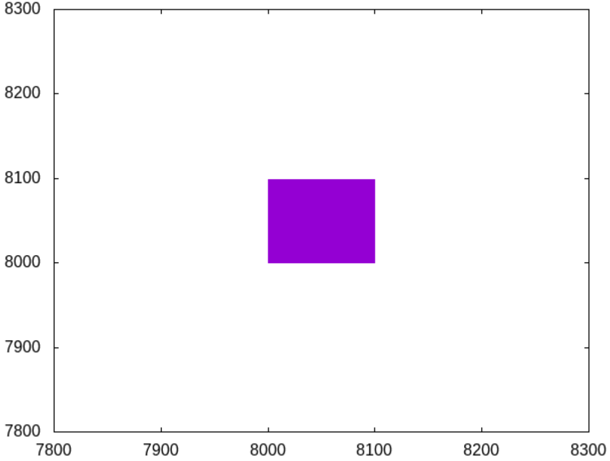

Figure: Dynamic on trees of size 10000.

Tree-decorated planar maps: combinatorial results.

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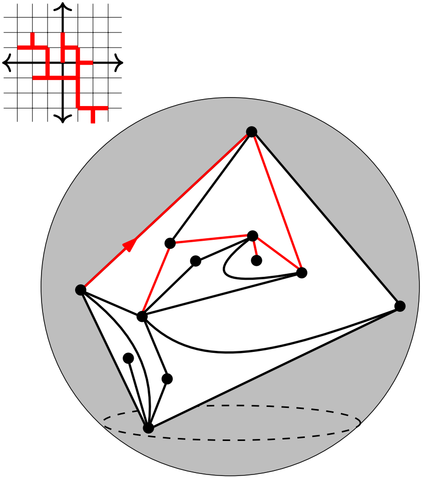
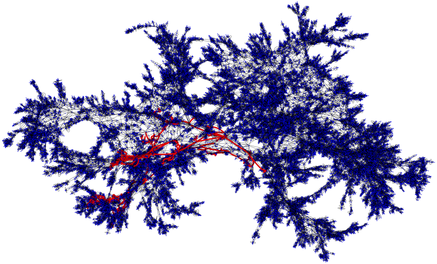


(a) tree-decorated quad. 10 faces, tree of size 6.



(b) Unif. tree-decorated quad. 90k faces and tree of size 500.

Interest

- **Physics:** They represent models of quantum gravity (models that "naturally" appear in an attempt to unify general relativity and quantum mechanics).
- **Combinatorics:** asymptotic growth, connectivity constants, simulations, etc. are easier to compute/simulate in these generalized lattices.
- **Probabilies:** Models of statistical mechanics where phase transitions are present, universality of limits (as in the central limit theorem) and where expected asymptotic behaviors can be exactly computed.

Combinatorics

Planar maps

Planar map: Proper drawing of a planar graph in the surface of the sphere...

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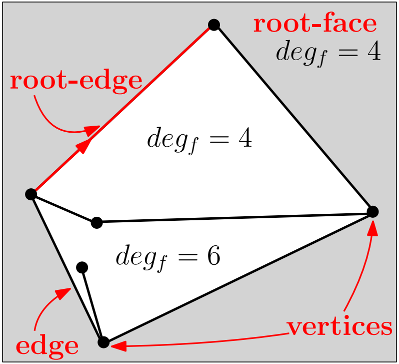
Planar maps

- A **face**= A connected component of the complement of the edges.
- The **root-edge**= distinguished half edge.
- The **root-face**= face to the left of the root-edge.
- **Degree of a face**= number of adjacent edges to it.

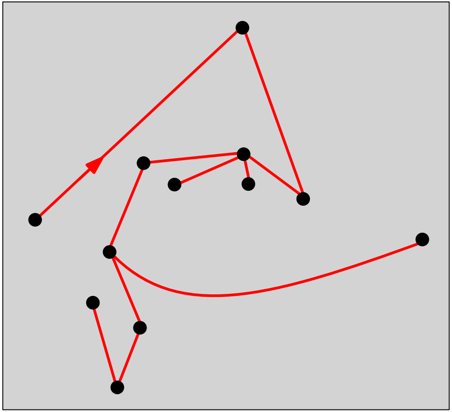


Planar trees

A **planar tree** is a rooted map with one face.

Number of planar trees with a edges

$$\mathcal{C}_a = \frac{1}{a+1}\binom{2a}{a}.$$

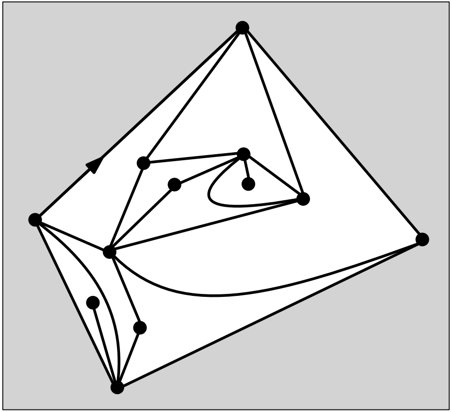


Quadrangulations

Quadrangulation: map whose faces have degree 4.

We know how to count them by analytic and bijective methods.

Analytic [Tutte '60] and Bijective [Cori-Vauquelin-Schaeffer '98].

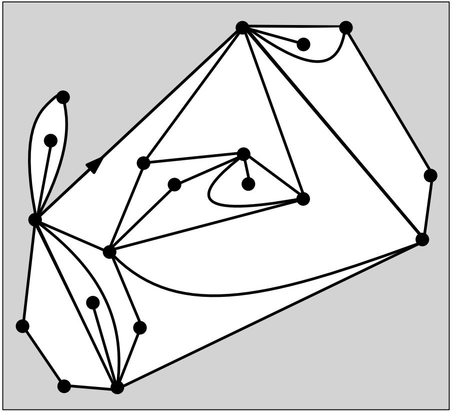


Quadrangulations with a boundary

Quadrangulation with a boundary: All faces, but the root-face, have degree 4.

We know how to count them with a boundary by analytic and bijective methods.

Analytic by [Bender & Canfield '94; Bouttier & Guitter '09] and bijective by [Schaeffer '97 ; Bettinelli '15]



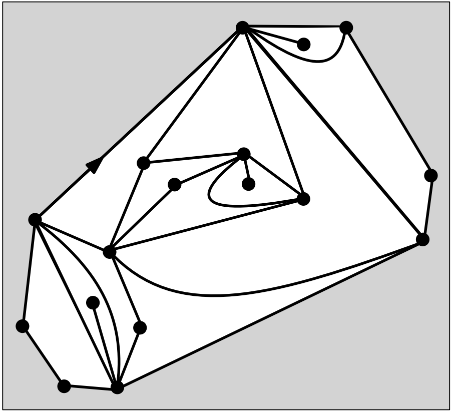
Quadrangulations with a **simple** boundary

Number of quadrangulations with a simple boundary with:

- f internal faces.
- **simple boundary** of size $2p$ (root-face of degree $2p$).

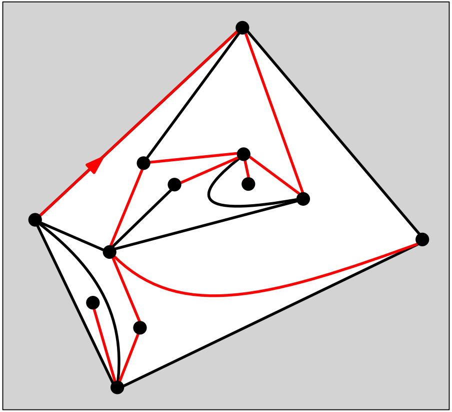
$$\frac{3^{f-p}2p}{(f+2p)(f+2p-1)}\binom{2f+p-1}{f-p+1}\binom{3p}{p}$$

Analytic [Bouttier & Guitter '09] and bijective [Bernardi & Fusy '17].



Spanning tree-decorated map (ST map): is a pair $(\mathfrak{m}, \mathfrak{t})$ where:

- $\mathfrak{m}$ is a rooted-map.
- $\mathfrak{t}$ is a submap of $\mathfrak{m}$ ($\mathfrak{t} \subset_M \mathfrak{m}$).
- $\mathfrak{t}$ is a spanning tree of $\mathfrak{m}$.



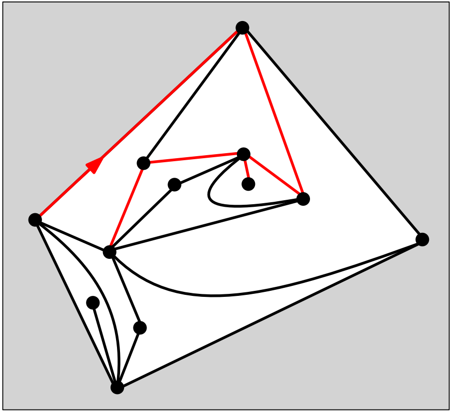
We know how to count ST maps by analytic and bijective methods.

Analytic by [Mullin '67] and bijective by [Walsh and Lehman '72; Cori, Dulucq & Viennot '86; Bernardi '06]

~~Spanning~~ tree-decorated maps

A (f, a) **tree-decorated map** is a pair $(\mathfrak{m}, \mathfrak{t})$ where:

- $\mathfrak{m}$ is a rooted map with f faces.
- $\mathfrak{t}$ is a submap of $\mathfrak{m}$ ($\mathfrak{t} \subset_M \mathfrak{m}$).
- $\mathfrak{t}$ is a tree with a edges.
- $\mathfrak{t}$ contains **the root-edge of $\mathfrak{m}$**.



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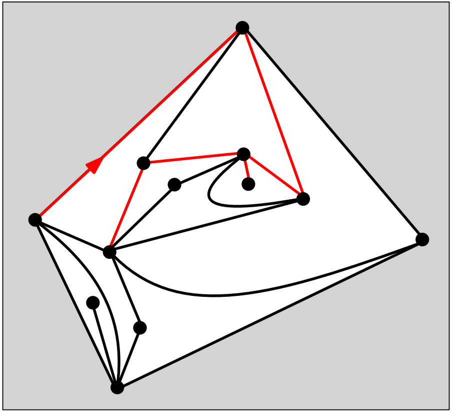


(f, a) **tree decorated maps interpolate:** In the case of quadrangulations

- $\underline{a = 1}$ $\rightarrow$ quadrangulations with f faces.
- $\underline{a = f + 1}$ $\rightarrow$ spanning-tree decorated quadrangulations with f faces.

Counting results

Theorem (F. & Sepúlveda '19)

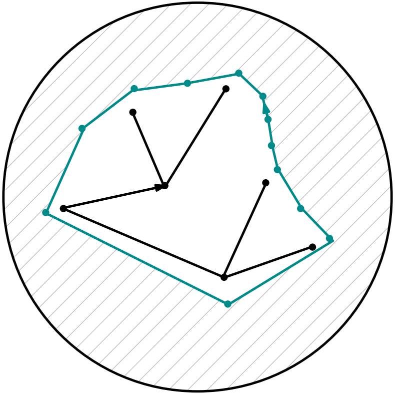
The number of (f, a) tree-decorated quadrangulations is

$$3^{f-a}\frac{(2f+a-1)!}{(f+2a)!(f-a+1)!}\frac{2a}{a+1}\binom{3a}{a,a,a}$$

Gluing bijection

Theorem (F. & Sepúlveda '19)

The set of (f, a) tree-decorated maps is in bijection with
(the set of maps with a simple boundary of size $2a$ and f interior faces)
$\times$ (the set of trees with a edges).



Probabilities

How to make sense of limits of graphs as their sizes go to infinity?

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We will explore two possible topologies:

- **Local topology:** elements are close if their balls (as maps) are equal up to a certain point.
- **Scaling limit topology:** comparison as metric spaces.

But how?

How to make sense of limits of graphs as their sizes go to infinity?
We will explore two possible topologies:

- **Local topology:** elements are close if their balls (as maps) are equal up to a certain point.
- **Scaling limit topology:** comparison as metric spaces.

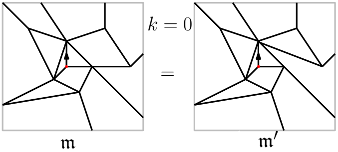
But how?

Graphs can be seen as metric spaces!
vertices + renormalized graph metric.

Topologies

- Local topology: local distance between two maps:

$$d_{loc}(\mathfrak{m}_1, \mathfrak{m}_2) = (1 + \sup\{r \geq 0 : B_r(\mathfrak{m}_1) = B_r(\mathfrak{m}_2)\})^{-1}$$



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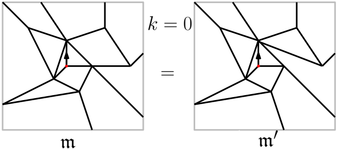
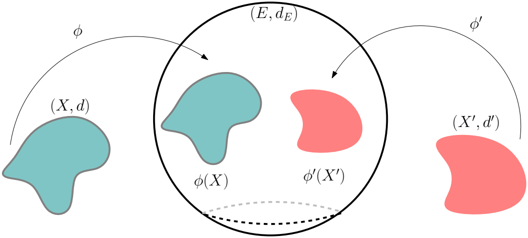


- Gromov-Hausdorff topology: Two metric spaces are close if there is a metric space in which both can be isometrically embedded such that the images are close.



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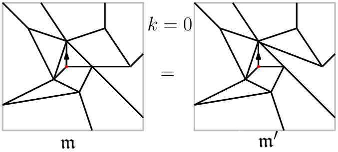

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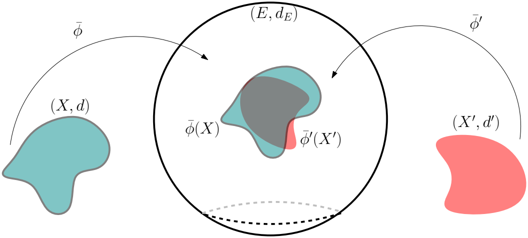


Uniform Trees

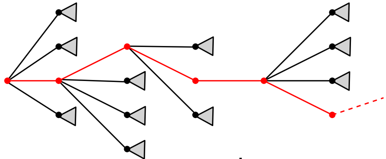
$\mathfrak{t}_{\mathbf{a}}$= Unif. tree with a edges.

Theorem (Kesten '86)

$$\mathfrak{t}_a \xrightarrow[local]{(d)} \mathfrak{t}_\infty$$

Properties

- $\mathfrak{t}_\infty$ *is an infinite tree.*
- *It has one infinite branch (the spine).*



$\mathfrak{t}_\infty$ **construction.**

Uniform Trees

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Theorem (Aldous '91)

$$\left(\mathfrak{t}_a, \frac{d_{\text{Tree}}}{a^{1/2}}\right) \xrightarrow[GH]{(d)} CRT$$

Properties

- *$\mathfrak{t}_\infty$ is an infinite tree.*
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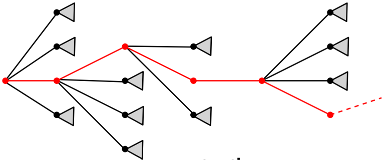
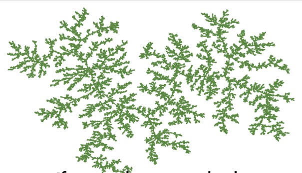
Properties

- *The CRT is a tree.*
- *Almost every point is a leaf.*
- *Hausdorff dimension 2.(Duquesne & Le Gall '05)*



$\mathfrak{t}_\infty$ construction.



Uniform random tree 50k edges.

Uniform quadrangulations

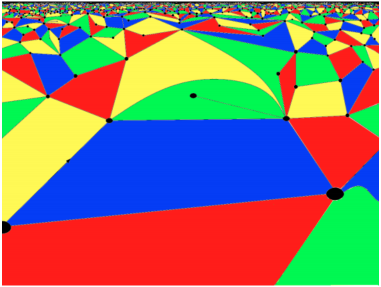
$\mathfrak{q}_{\mathbf{f}}$= Unif. quadrangulation with f faces.

Theorem (Krikun '06)

$$\mathfrak{q}_f \xrightarrow[local]{(d)} UIPQ$$

Properties

- *The UIPQ is an infinite quad.*



(Sketch by N. Curien)

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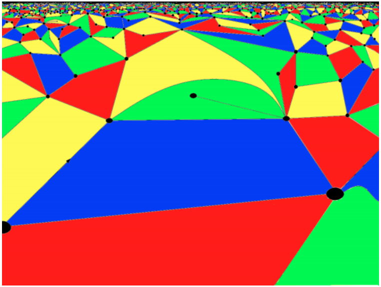


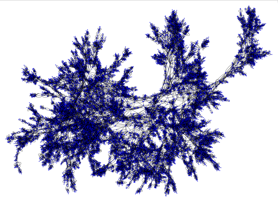
(Sketch by N. Curien)

Theorem (Miermont '13, Le Gall '13)

$$\left(\mathfrak{q}_f, \frac{\mathrm{d_{map}}}{f^{1/4}}\right) \xrightarrow[GH]{(d)} \textit{Brownian map}$$

Properties

- *Hausdorff dim. is 4 (Le Gall '07).*
- *Homeomorphic to $\mathbb{S}^2$ (Le Gall & Paulin '08).*



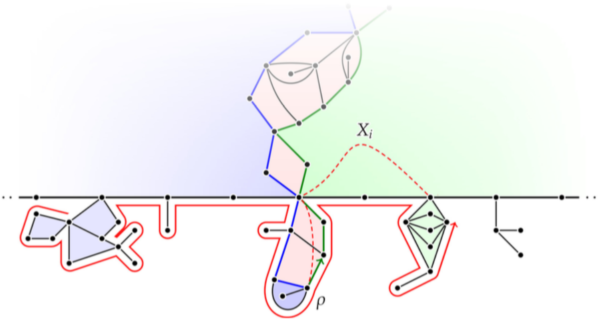
Unif. quadrangulation 30k faces.

Unif. quad. with a boundary: local limit

$\mathfrak{q}_{\mathbf{f,p}}$= Unif. quadrangulations with a boundary of size $2p$ and f faces.

Theorem (Curien & Miermont '12)

$$\mathfrak{q}_{f,p} \xrightarrow[local(f\to\infty)]{(d)} \mathfrak{q}_{\infty,p} \xrightarrow[local(p\to\infty)]{(d)} UIHPQ$$



UIHPQ (sketch by N. Curien & A. Caraceni)

Uniform quadrangulation with a boundary: GH limit

$\mathfrak{q}_{\mathbf{f},\mathbf{p}}$= Unif. quadrangulations with a boundary of size $2p$ and f faces. For a sequence $(p(f))_{f\in\mathbb{N}}$, define $\overline{p} = \lim p(f)f^{-1/2}$ as $f \to \infty$.

Theorem (Scaling limit (Bettinelli '15))

$$\left(\mathfrak{q}_{f,p(f)}, \frac{\mathrm{d}_{\mathrm{map}}}{s(f,p(f))}\right) \xrightarrow[GH]{(d)} \begin{cases} \textit{Brownian map} & \textit{if } s(f,p(f)) = f^{1/4} \textit{ and } \overline{p} = 0 \\ \textbf{\textit{Brownian disk}} & \textit{if } s(f,p(f)) = f^{1/4} \textit{ and } \overline{p} \in (0,+\infty) \\ \textit{CRT} & \textit{if } s(f,p(f)) = 2p(f)^{1/2} \textit{ and } \overline{p} = \infty \end{cases}$$

Uniform quadrangulation with a boundary: GH limit

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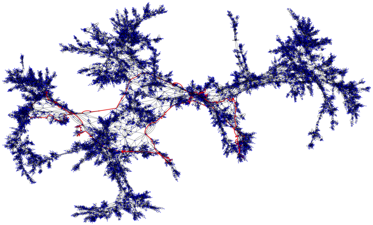
Properties (Bettinelli & Miermont '15)

Brownian disk properties

- *The boundary is simple.*
- *Hausdorff dim. 4 in the interior, 2 in the boundary.*
- *Homeomorphic to the disk 2d.*



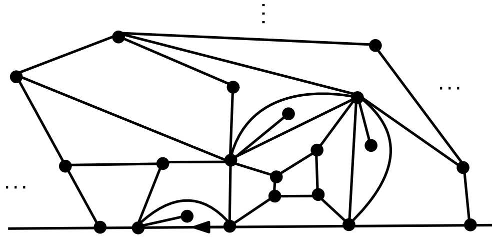
Unif. quad. with 30k interior faces and boundary 173.

Unif. quad. with a simple boundary: local limit

$\mathfrak{q}^{\mathbf{S}}_{\mathbf{f,p}}$= Unif. quadrangulations with a **simple** boundary of size $2p$ and f faces.

Theorem (Curien & Miermont '12)

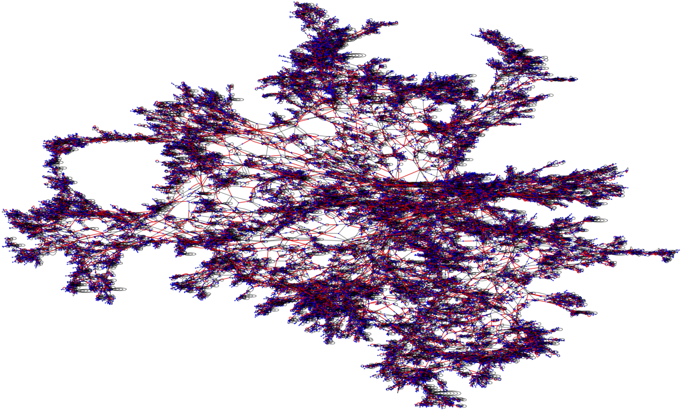
$$\mathfrak{q}^{S}_{f,p} \xrightarrow[local(f\to\infty)]{(d)} \mathfrak{q}^{S}_{\infty,p} \xrightarrow[local(p\to\infty)]{(d)} UIHPQ^{S}$$



sketch of a $UIHPQ^{S}$.

Uniform ST map

- Convergence for the local topology (Sheffield '11).
- The limit (if it exists) seems not to the Brownian map.
- Expected diameter is of order n^{χ} for $0.275 \leq \chi \leq 0.288$ (Ding & Gwynne '18, Gwynne, Holden & Sun '16).



Uniform ST map 100k edges.

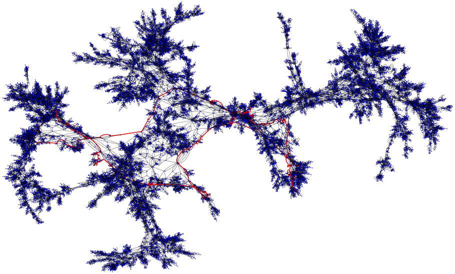
Uniform quadrangulation with a **simple** boundary: GH limit

$\mathfrak{q}_{\mathbf{f,p}}{}^S =$ Unif. quadrangulations with **simple** boundary $2p$ and f faces.
For a sequence $(p(f))_{f\in\mathbb{N}}$, define $\bar{p} = \lim p(f) f^{-1/2}$ as $f \to \infty$.

Theorem (Scaling limit (Bettinelli, Curien, F., Sepúlveda '20+))

If $\bar{p} \in (0, +\infty)$, then

$$\left(\mathfrak{q}_{f,p(f)}{}^S, \frac{d_{\mathrm{map}}}{f^{1/4}}\right) \xrightarrow[GH]{(d)} \textit{Brown. disk}$$

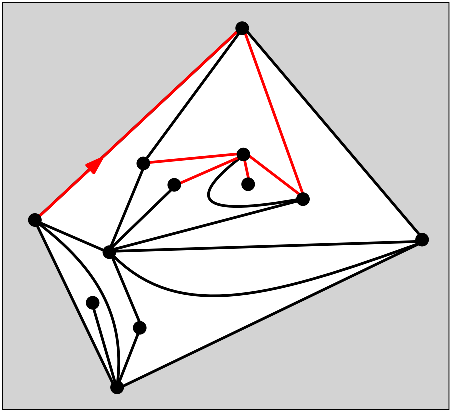


Unif. quad. with 30k interior faces and boundary 173.

Uniform tree-decorated maps

$\mathfrak{q}_{\mathbf{f}}^{\mathbf{a}}$= Unif. tree-decorated map with f faces and a tree of size a.

Why it is interesting to study this family??



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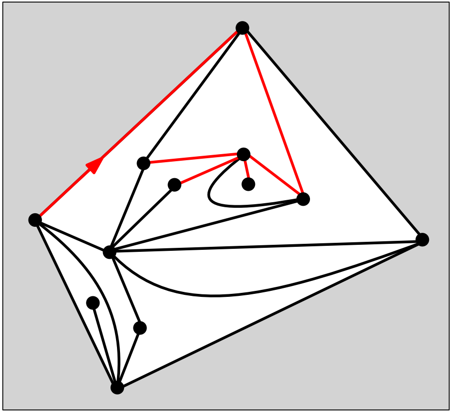

- **New statistical mechanic family**

$\mathbb{P}(\mathfrak{q}_f^a = (\mathfrak{m}, \cdot)) \propto \#\{\text{trees of size } a \text{ in } \mathfrak{m}\}$

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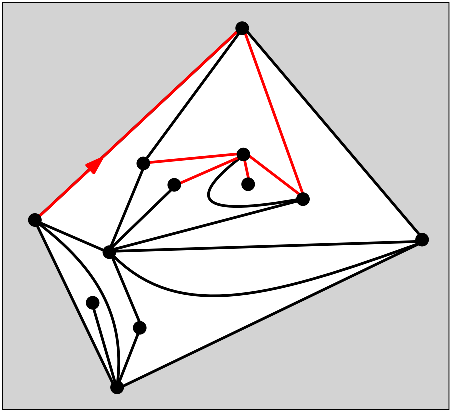


- **New statistical mechanic family**

$\mathbb{P}(\mathfrak{q}_f^a = (\mathfrak{m}, \cdot)) \propto \#\{\text{trees of size } a \text{ in } \mathfrak{m}\}$

- **It interpolates**
  - $a = 1$= Uniform quadrangulations.
  - $a = f + 1$= Uniform ST quadrangulations.

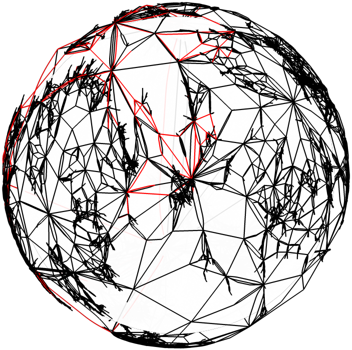
Local limit results

$\mathfrak{q}_f^a$= Unif. tree-decorated map with f faces and a tree of size a.

Theorem (F. & Sepúlveda '19+)

$$\mathfrak{q}_f^a \xrightarrow[local, f\to\infty]{(d)} \mathfrak{q}_\infty^a \xrightarrow[local, a\to\infty]{(d)} \mathfrak{q}_\infty^\infty$$

$\mathfrak{q}_\infty^\infty$ is the "gluing" of $\mathfrak{t}_\infty$ and *$UIHPQ^S$*.



Scaling limit results

$\mathfrak{q}_{\mathbf{f}}^{\mathbf{a}}$= Unif. tree-decorated map with f faces and a tree of size a.

Corollary (F. & Sepúlveda '19+)

Let $\mathfrak{q}_f^{a(f)} = (\mathfrak{q}, \mathfrak{t})$, *with* $a(f) \leq f + 1$. *Then as* $a(f) \to \infty$,

$$\left(\mathfrak{t}, \frac{\mathrm{d}_{\mathrm{Tree}}}{a(f)^{1/2}}\right) \xrightarrow[GH]{(d)} CRT.$$

Scaling limit conjecture

$\mathfrak{q}_{\mathbf{f}}^{\mathbf{a}}$= Unif. tree-decorated map with f faces and a tree of size a.

Conjecture (F. & Sepúlveda '19+)

Let $a(f) = O(f^{\alpha})$. Depending on α as $f \to \infty$

$$\left(\mathfrak{q}_f^{a(f)}, \frac{\mathrm{d_{map}}}{f^{\beta}}\right) \xrightarrow[GH]{(d)} \begin{cases} \textit{Brownian map} & \textit{if } \alpha < 1/2, \beta = 1/4 \textit{(Proved)} \\ \textbf{\textit{Shocked map}} & \textit{if } \alpha = 1/2, \beta = 1/4 \textit{(In progress)} \\ \textit{Tree-decorated map} & \textit{if } \alpha > 1/2, \\ & \quad \beta = \left(2\chi - \frac{1}{2}\right)\alpha - \chi + \frac{1}{2} \end{cases}$$

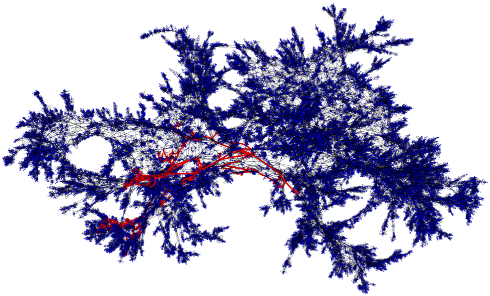
Scaling limit conjecture

q_f^a= Unif. tree-decorated map with f faces and a tree of size a.

Conjecture (F. & Sepúlveda '19+)

Let $a(f) = O(f^\alpha)$. Depending on α as $f \to \infty$

$$\left(q_f^{a(f)}, \frac{d_{map}}{f^\beta}\right) \xrightarrow[GH]{(d)} \begin{cases} \textit{Brownian map} & \text{if } \alpha < 1/2, \beta = 1/4 \textit{(Proved)} \\ \textbf{\textit{Shocked map}} & \text{if } \alpha = 1/2, \beta = 1/4 \textit{(In progress)} \\ \textit{Tree-decorated map} & \text{if } \alpha > 1/2, \\ & \beta = \left(2\chi - \frac{1}{2}\right)\alpha - \chi + \frac{1}{2} \end{cases}$$



Shocked map

Shocked map properties:

- **It is not degenerated** (Proved).
- It should be the gluing of a Brownian disk and a CRT.
- Hausdorff dim. 4 (Proved).
- The tree has Hausdorff dim. 2 (In progress, ≤ 2 proved).
- Homeomorphic to $\mathbb{S}^2$. (Proved).

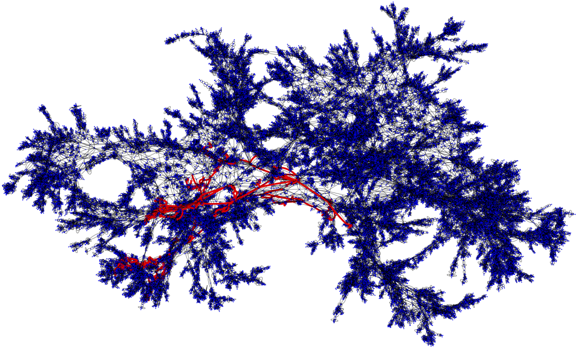


Figure: **Unif. (90k,500) tree-decorated quadrangulation.**

Why shocked?

Thanks for your attention!

Remarks and extensions

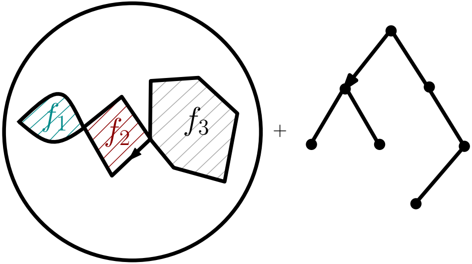
The bijection makes a correspondence between:

[Tree-decorated map]		**[Map with a boundary, Tree]**
Faces of degree q	$\longleftrightarrow$	Internal faces of degree q
Internal vertices of degree d	$\longleftrightarrow$	Internal vertices of degree d
Internal edges	$\longleftrightarrow$	Internal edges
Corner of the tree	$\longleftrightarrow$	Boundary vertices.

- We can restrict the bijection to q-angulations.
- It can be restricted to some subfamilies of trees:
  1. Binary tree-decorated Maps.
  2. SAW decorated maps (Already done by Caraceni & Curien).

What do we obtain when the boundary is not simple?

For bridgeless maps it gives BUBBLE-MAPS!



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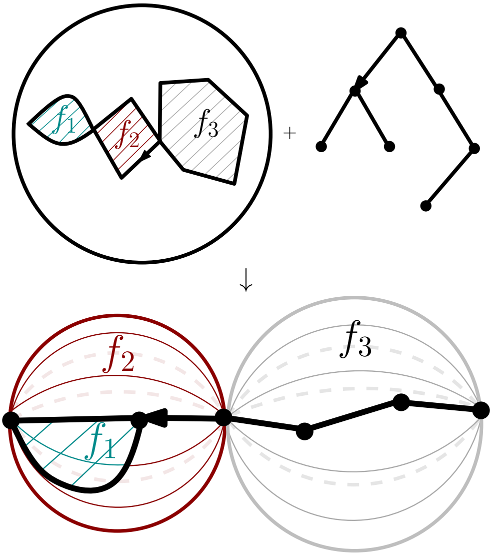

Counting results

In the case of spanning tree decorated quadrangulations rooted in the tree we obtain

$$\mathcal{C}_{2,f} = \frac{2}{(f+1)(f+2)}\binom{3f}{f,f,f}$$

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A possible generalization of Catalan numbers:

$$\mathcal{C}_{m,n} = m!\left(\prod_{i=1}^{m}\frac{1}{(n+i)}\right)\binom{(m+1)n}{\underbrace{n,n,\dots,n}_{m+1 \text{ times}}} = \binom{m+n}{n}^{-1}\binom{(m+1)n}{\underbrace{n,n,\dots,n}_{m+1 \text{ times}}}$$

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Proposition

$\mathcal{C}_{m,n}$ is an integer $\forall n, m$.

Proof by D. Sénizergues.

Define $A_{n,m} = \#$ standard young tableaux of shape $\lambda = \underbrace{(n, n, ..., n)}_{m \text{ times}}$.

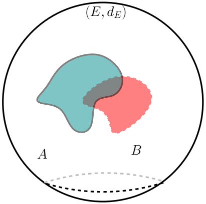
From the hook-length formula we see that

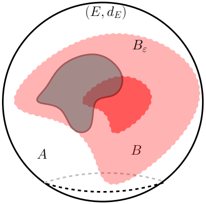
$$C_{m,n} = \left(\prod_{i=1}^{m-1} \binom{n+i}{i} \right) \times A_{n,m+1}$$

$\square$

Let (E, d_E) be a metric space and $A, B \subset E$. The **Hausdorff distance** is

$$d_H(A, B) = \inf\left\{\varepsilon > 0 : A \subset B_\varepsilon, B \subset A_\varepsilon\right\}$$





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Gromov-Hausdorff topology

Consider the set S of compact metric spaces up to isometry classes. The **Gromov-Hausdorff distance** between two metric spaces (X, d) and (X', d') is defined as

$$\mathrm{d_{GH}}((\mathrm{X}, \mathrm{d}), (\mathrm{X}', \mathrm{d}')) = \inf \mathrm{d_H}(\phi(\mathrm{X}), \phi'(\mathrm{X}'))$$

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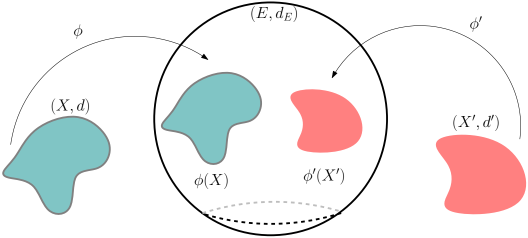

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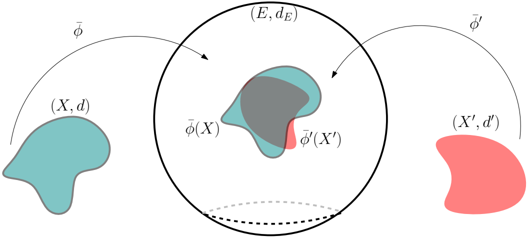

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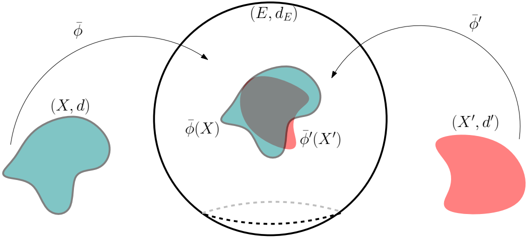

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Proposition

The function $\mathrm{d_{GH}}$ *induces a metric on* S*. The space* $(S, \mathrm{d_{GH}})$ *is separable and complete.*

Convergence in distribution

We say that:

$$X_n \xrightarrow[top]{(d)} X$$

if for any continuous bounded function $f : top \to \mathbb{R}$

$$\mathbb{E}(f(X_n)) \to \mathbb{E}(f(X))$$

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In the case of the local topology it translates into:
for all $r \in \mathbb{R}$, there exists $N_0 \in \mathbb{N}$, such that for any $n \geq N_0$

$$\mathbb{P}(B_r(X_n) = \mathfrak{m}) = \mathbb{P}(B_r(X) = \mathfrak{m})$$

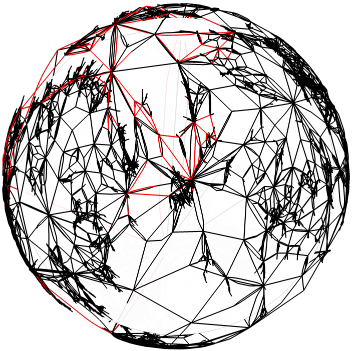
Local limit results

$\mathfrak{q}_{\mathbf{f}}^{\mathbf{a}}$= Unif. tree-decorated map with f faces and a tree of size a.

Theorem (F. & Sepúlveda '19+)

$$\mathfrak{q}_f^a \xrightarrow[local, f\to\infty]{(d)} \mathfrak{q}_\infty^a \xrightarrow[local, a\to\infty]{(d)} \mathfrak{q}_\infty^\infty$$

$\mathfrak{q}_\infty^\infty$ is the "gluing" of $\mathfrak{t}_\infty$ and $UIHPQ^S$.



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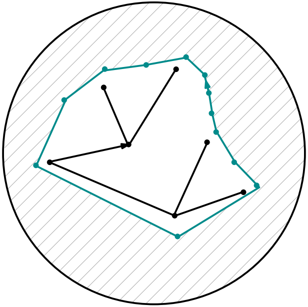

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