Some models at the interface of probability theory and combinatorics: particle systems and maps.

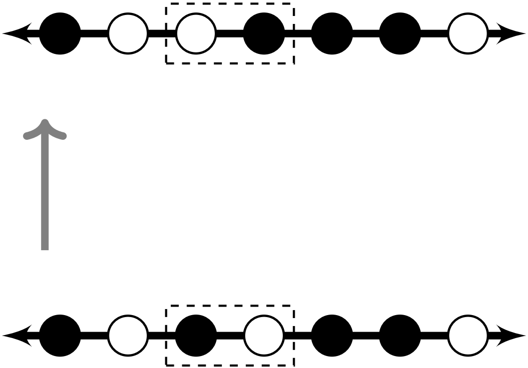
Luis Fredes

PhD defense.
Under the supervision of J.F. Marckert.

1. Invariant measures of discrete interacting particles systems: algebraic aspects.
2. Survival and coexistence for spatial population models with forest fire epidemics.
3. Tree-decorated planar maps: combinatorial results.
4. Tree-decorated planar maps: local limits.

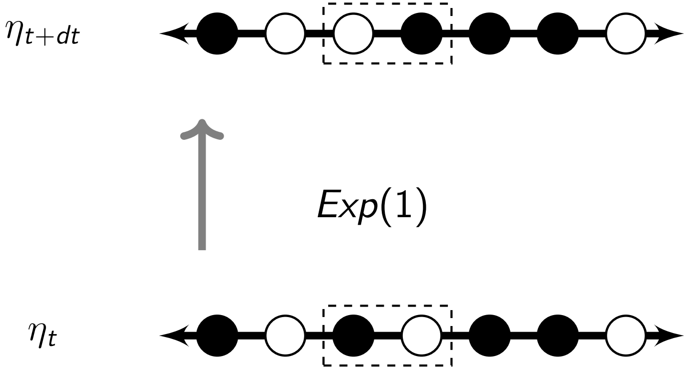
Example: **TASEP**

η_{t+dt}

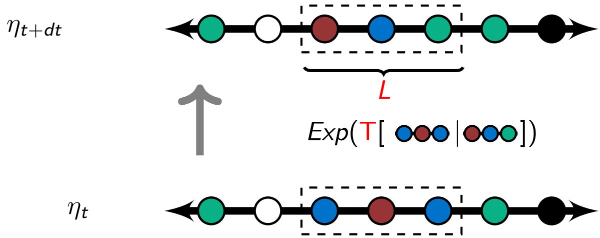


η_t

Example: TASEP



Particle systems of our interests



Our setting: model depends on 4 parameters:

- Graph $G = (V, E)$ belonging to $\mathbb{Z}^d$, $\mathbb{Z}/n\mathbb{Z}$.
- Set of $\kappa \in \mathbb{N} \cup \{\infty\}$ colors

$$E_\kappa = \{0, 1, \dots, \kappa - 1\}.$$

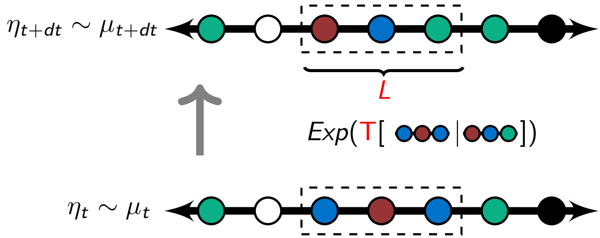
- Dependence neighborhood $L \geq 2$.
- Jump rate matrix

$$\mathrm{T} = [\mathrm{T}_{[\boldsymbol{u}|\boldsymbol{w}]}]_{\{\boldsymbol{u},\boldsymbol{w} \in E_\kappa^L\}}.$$

Notation:

$$\boldsymbol{\eta_t(v)} \rightarrow \text{ color of vertex } \boldsymbol{v} \text{ at time } \boldsymbol{t}.$$

Particle systems of our interests



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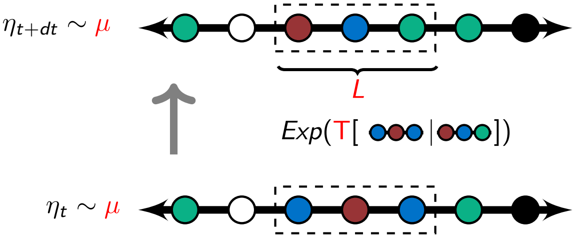
Invariant measure

Definition

Given T, a distribution μ is said to be *invariant* if $\eta_t \sim \mu$ for any $t \geq 0$, when $\eta_0 \sim \mu$.



Some references:

- Well definition of PS given T:
  1. $\kappa < \infty$, always well defined. **[Liggett '85]**.
  2. $\kappa = \infty$, not always. Some techniques:
     - Graphic method **[Harris 72']**.
     - Functional analysis **[Liggett '73]**.
     - Stochastic domination **[Andjel '82]**.
- Existence of invariant measures for specific PS . **[Andjel '82]**.
- Computation of invariant measures for specific PS. **[Derrida et al '93, Blythe & Evans '07...]**.
- Uniqueness / ergodicity?
  Convergence to the invariant measure for specific PS?
  Rate of convergence? **[Benjamini et al '05, Labbé & Lacoin '16...]**.

Our main question:

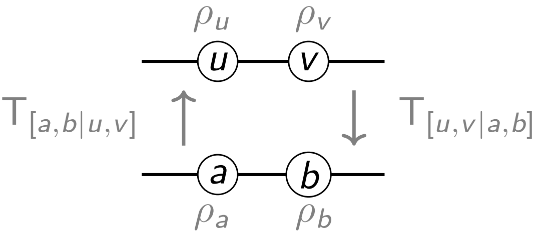
Given a (class of) measure, is it possible to characterize the T's for which this measure is invariant?

- A classical sufficient condition for invariance of **product measures**

Detailed balance equations

The **product measure** $\rho^{\mathbb{Z}}$ is invariant by T on $\mathbb{Z}$ if

$$\rho_a \rho_b \mathrm{T}_{[a,b|u,v]} = \rho_u \rho_v \mathrm{T}_{[u,v|a,b]} \qquad \forall a, b, u, v \in E_\kappa$$



- The **product measure** case is partially known. **[Fajfrová et al '16]**.

Our main question:

Is it possible to characterize the T for which the distribution of a Markov chain is invariant ?"

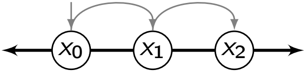
Markov Distribution

A process X has a Markov distribution (ρ, M), with Markov Kernel (MK) M of memory $m = 1$ and initial distribution ρ, if for any $\boldsymbol{x} \in E_{\kappa}^{n+1}$

$$\mathbb{P}(X[\![0, n]\!] = \boldsymbol{x}) = \rho_{x_0} \prod_{j=0}^{n-1} M_{x_j, x_{j+1}}.$$

$$\mathbb{P}(X[\![0, 2]\!] = \boldsymbol{x}) =$$

$$\rho_{x_0} M_{x_0, x_1} M_{x_1, x_2}$$



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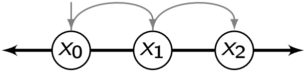
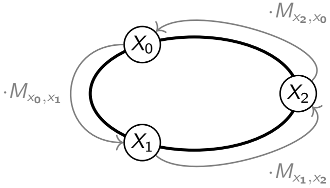


Gibbs Distribution

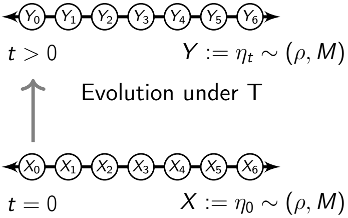
A vector $(X_k, k \in \mathbb{Z}/n\mathbb{Z})$ is said to have a Gibbs distribution $G(M)$ characterized by a MK M, if for any $\boldsymbol{x} \in E_\kappa^n$,

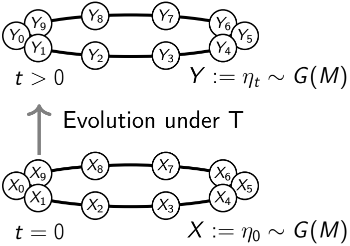
$$\mathbb{P}(X[\![0,n-1]\!] = \boldsymbol{x}) = \frac{\prod_{j=0}^{n-1} M_{x_j, x_{j+1} \mod n}}{\mathrm{Trace}(M^n)}.$$

$$\mathbb{P}(X[\![0,2]\!] = \boldsymbol{x}) \propto$$



Invariance schemes





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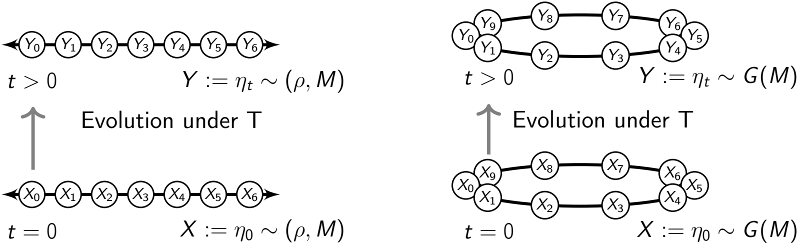

Main Theorem (F. & Marckert '17)

Let κ be finite, $L = 2$ and $m = 1$. If $M > 0$ (strictly positive entries) then the following statements are equivalent:

1. (ρ, M) is invariant by T on $\mathbb{Z}$.
2. $G(M)$ is invariant by T on $\mathbb{Z}/7\mathbb{Z}$.
3. $G(M)$ is invariant by T on $\mathbb{Z}/n\mathbb{Z}$, for all $n \geq 3$.

Suppose $\mu_t = (\rho, M)$. We define

$$\mathsf{Line}_n^{M,\mathsf{T}}(\boldsymbol{x}) := \frac{\partial}{\partial t}\mu_t(x_1 x_2 \ldots x_{n-1} x_n)$$

$=$ Mass creation rate of x

$-$ Mass destruction rate of x

Definition

A (ρ, M) MD under its invariant distribution is said to be **invariant by** T on the line when $\mathsf{Line}_n^{M,T} \equiv 0$, for all $n \in \mathbb{N}$.

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$$\mathrm{Line}_n^{M,\mathsf{T}}(\boldsymbol{x}) := \frac{\partial}{\partial t}\mu_t(x_1 x_2 \dots x_{n-1} x_n)$$

$$= \sum_{\substack{x_{-1}, x_0, \\ x_{n+1}, x_{n+2} \in E_\kappa}} \sum_{j=0}^{n} \sum_{(u,v)\in E_\kappa^2} \left(\mathsf{T}_{[u,v|x_j,x_{j+1}]} \Big(\rho_{x_{-1}} \prod_{\substack{-1\le k\le n+1 \\ k\notin\{j-1,j,j+1\}}} M_{x_k,x_{k+1}} \Big) M_{x_{j-1},u} M_{u,v} M_{v,x_{j+2}} \right.$$
$$\left. - \mathsf{T}_{[x_j,x_{j+1}|u,v]} \Big(\rho_{x_{-1}} \prod_{k=-1}^{n+1} M_{x_k,x_{k+1}} \Big) \right)$$

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Elements of the proof: algebraization

Suppose $\mu_t = (\rho, M)$. We define

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$$\underbrace{\left(\left(\sum_{(u,v)\in E_\kappa^2} \mathrm{T}_{[u,v|x_j,x_{j+1}]} \frac{M_{x_{j-1},u} M_{u,v} M_{v,x_{j+2}}}{M_{x_{j-1},x_j} M_{x_j,x_{j+1}} M_{x_{j+1},x_{j+2}}}\right) - \mathrm{T}^{\mathrm{out}}_{[x_j,x_{j+1}]}\right)}_{Z_{x_{j-1},x_j,x_{j+1},x_{j+2}}}$$

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A (ρ, M) MD under its invariant distribution is said to be **invariant by** T on the line when $\mathrm{Line}_n^{M,T} \equiv 0$, for all $n \in \mathbb{N}$.

Elements of the proof: algebraization

Suppose $\mu_t = (\rho, M)$. We define **for** $M > 0$

$$\mathrm{NLine}_n^{M,\mathsf{T}}(\boldsymbol{x}) := \frac{\mathrm{Line}_n^{M,\mathsf{T}}(\boldsymbol{x})}{\prod_{i=1}^{n-1} M_{x_i, x_{i+1}}}$$

$$= \sum_{\substack{x_{-1}, x_0, \\ x_{n+1}, x_{n+2} \in E_\kappa}} \sum_{j=0}^{n} \Big(\rho_{x_{-1}} M_{x_{-1}, x_0} M_{x_0, x_1} \quad \times \quad Z_{x_{j-1}, x_j, x_{j+1}, x_{j+2}} \quad \times \quad M_{x_n, x_{n+1}} M_{x_{n+1}, x_{n+2}} \Big)$$

Definition

A (ρ, M) MD under its invariant distribution is said to be **invariant by** T on the line when $\mathrm{Line}_n^{M,T} \equiv 0$, for all $n \in \mathbb{N}$.

Important message:

- **Line:** If $M > 0$, then (ρ, M) **is invariant by** T **on** $\mathbb{Z}$ $\iff$

$$\mathrm{NLine}_n^{M,\mathrm{T}}(x) = Z^L_{x_{\mathbf{1}},x_{\mathbf{2}},x_{\mathbf{3}}} + \sum_{j=2}^{n-2} Z_{x_{j-\mathbf{1}},x_j,x_{j+\mathbf{1}},x_{j+\mathbf{2}}} + Z^R_{x_{n-\mathbf{2}},x_{n-\mathbf{1}},x_n} = 0 \qquad \textbf{for all } n \in \mathbb{N}$$

- **Cycle of length n:** If $M > 0$, then $G(M)$ **is invariant by** T **on** $\mathbb{Z}/7\mathbb{Z}$ $\iff$

$$\mathrm{NCycle}_n^{M,\mathrm{T}}(x) = \sum_{j=0}^{n-1} Z_{x_{\overline{j-\mathbf{1}}},x_j,x_{\overline{j+\mathbf{1}}},x_{\overline{j+\mathbf{2}}}} = 0$$

where $\bar{i} := i \mod n$

Sketch of proof

1) $\implies$ 2) in main theorem: $M > 0$

$$\textbf{1)}(\rho, M) \text{ is invariant by T on } \mathbb{Z} \implies \textbf{2)} G(M) \text{ is invariant by T on } \mathbb{Z}/7\mathbb{Z}$$

$$\Updownarrow$$

$$\text{NLine}_n^{M,T} \equiv 0 \quad \forall n \in \mathbb{N} \implies \text{NCycle}_7^{M,T} \equiv 0$$

Consider $\boldsymbol{x} \in E_\kappa^7$ and $\boldsymbol{w} = \underbrace{\boldsymbol{x} \ldots \boldsymbol{x}}_{\ell \text{ times}}$

$$\text{NLine}_{7\ell}^{M,T}(\boldsymbol{w}) = \underbrace{\text{Bound. terms}}_{O(1)} + \ldots \qquad = 0$$

$$\leftarrow x_1 - x_2 - x_3 - x_4 - x_5 - x_6 \cdots\cdots x_2 - x_3 - x_4 - x_5 - x_6 - x_7 \rightarrow$$

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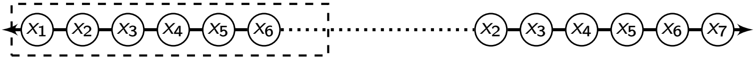

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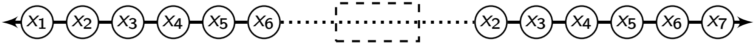

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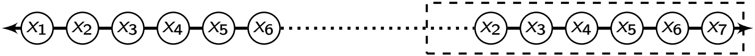


Theorem- Strongest form (F. & Marckert '17)

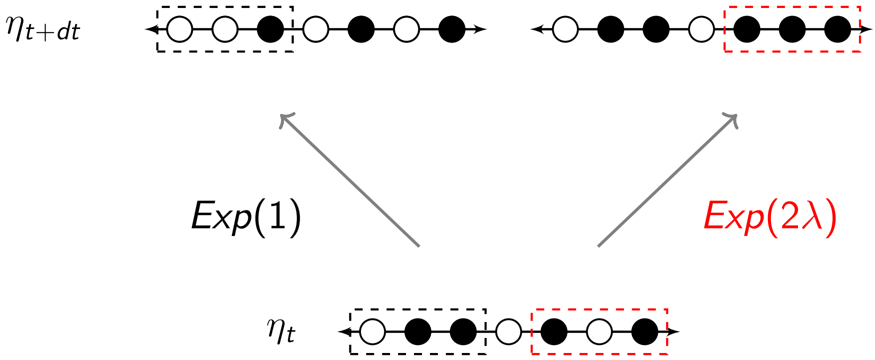
Let E_κ be finite, $L \geq 2$, $m \in \mathbb{N}$. If $M > 0$ (strictly positive entries) then the following statements are equivalent:

1. (ρ, M) is invariant by T on $\mathbb{Z}$.
2. $G(M)$ is invariant by T on $\mathbb{Z}/h\mathbb{Z}$.
3. $G(M)$ is invariant by T on $\mathbb{Z}/n\mathbb{Z}$, for all $n \geq m + L$.

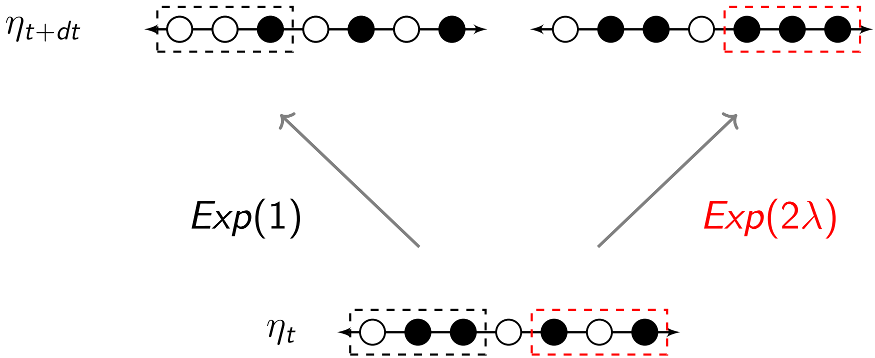
with $h := 4m + 2L - 1$

The system of equations in 2) is finite, of degree h in M and linear in T.

Application: the contact process



Application: the contact process



Corollary (F. & Marckert '17)

The contact process does not have a non-trivial MD of any memory $m \geq 0$ as invariant distribution.

Algorithm

We give an algorithm to compute the set of $M > 0$ invariant by a given T. Case **finite number of colors ($\kappa < \infty$), memory 1 ($m = 1$) and range 2 ($L = 2$)**.

- Find the set of all ν satisfying **(linear algebra)**

$$\sum_{u,v \in E_\kappa} \left(\nu_{c,u,v} \mathsf{T}_{[u,v|a,b]} + \nu_{a,u,v} \mathsf{T}_{[u,v|b,c]} + \nu_{b,u,v} \mathsf{T}_{[u,v|c,a]} \right)$$
$$= \nu_{a,b,c} \left(\mathsf{T}^{\text{out}}_{[a,b]} + \mathsf{T}^{\text{out}}_{[b,c]} + \mathsf{T}^{\text{out}}_{[c,a]} \right).$$

- Property: For each ν there exists at most one M satisfying

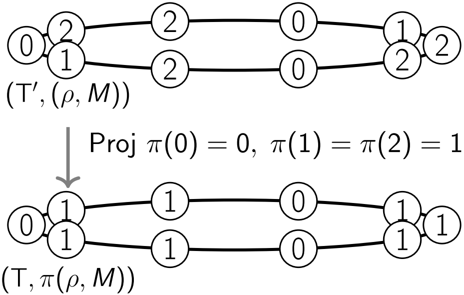
$$\nu_{x,y,z} = \frac{M_{x,y} M_{y,z} M_{z,x}}{\text{Trace } M^3}$$

  When M exists, compute it **(algebra)**.
- Test if M satisfies $\text{NCycle}_7^{M,\mathsf{T}} \equiv 0$ **(linear algebra/Gröbner)**.

Corollary

There exists a jump rate matrix T *which possesses some hidden Markov chain as invariant distribution.*

Idea:



$$\mathsf{T}_{[0,0,0|0,1,0]} = 270, \quad \mathsf{T}_{[0,1,0|0,0,0]} = 294.$$

Invariant product measures

Detailed balance equations (DBE)

The product measure $\rho^{\mathbb{Z}}$ is invariant by T on $\mathbb{Z}$ if

$$\rho_a\rho_b \mathsf{T}_{[a,b|u,v]} = \rho_u\rho_v \mathsf{T}_{[u,v|a,b]} \qquad \forall a,b,u,v \in E_\kappa$$

Theorem (F. & Marckert '17)

Let $\kappa < \infty$ and $L = 2$. If ρ is a measure with support E_κ then the following are equivalent:

1. The product measure $\rho^{\mathbb{Z}}$ is invariant by T on $\mathbb{Z}$.
2. The product measure $\rho^{\mathbb{Z}/3\mathbb{Z}}$ is invariant by T on $\mathbb{Z}/3\mathbb{Z}$.

$$\begin{aligned} Z_{\cancel{a},b,c,\cancel{d}} &= \sum_{u,v\in E_\kappa} \left(\frac{\rho_u\rho_v\cancel{\rho_d}}{\rho_b\rho_c\cancel{\rho_d}} \mathsf{T}_{[u,v|b,c]} - \mathsf{T}_{[b,c|u,v]} \right) \\ &= \sum_{u,v\in E_\kappa} \frac{1}{\rho_a\rho_b} \left(\rho_u\rho_v \mathsf{T}_{[u,v|b,c]} - \rho_b\rho_c \mathsf{T}_{[b,c|u,v]} \right) \end{aligned}$$

2) $\iff$ $Z_{a,b} + Z_{b,c} + Z_{c,a} = 0, \ \forall a,b,c \in E_\kappa$ and **(DBE)** $\implies$ $Z \equiv 0$.

Survival and coexistence for spatial population models with forest fire epidemics. with A. Linker. and D. Remenik.

Motivation from math-biology: Find models of population dynamics achieving biodiversity.

References:

- Predators **[Mimura & Kan-on '86, Hofbauer & Sigmund '89, Schreiber '97.]**
- Random fluctuations in the environment.**[Mao, Marion & Renshaw '02, Zhu and Yin '09.]**
- Random diseases.**[Holt & Pickering '85, Saenz & Hethcote '06.]**
- Crowding effect.**[Sevenster '96, Hartley & Shorrocks '02, Gavina et al '18.]**

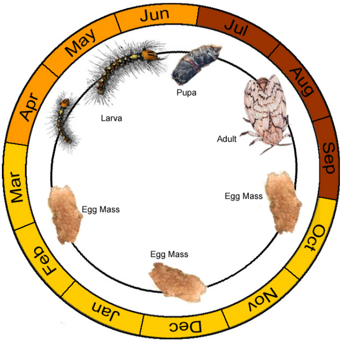
Our contribution: Design + study of a stochastic model of population dynamics with long-time coexistence.

Gypsy moth infestation model.

- One year life's cycle. **Discrete time.**
- When population density is too high, it gets attacked by epidemics. **Forest fires.**

Durrett & Remenik '09.

Our work: Multi-type model



Ohio Department of Agriculture

Our Multi-type model:

- **Space:** Graph $G_N = (V_N, E_N)$ on N vertices.
- **Configurations:** $\eta_k = \{\eta_k(v)\}_{v \in V_N}$ global state of the system at time k.

$$\boldsymbol{\eta_k(v)} \rightarrow \text{type of (the particle at) } \boldsymbol{v} \text{ at time } \boldsymbol{k}.$$

- **Initial configuration:** η_0.

Evolution: Two consecutive steps (per unit of time).

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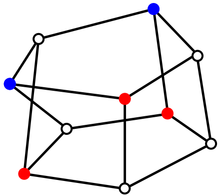
- **Growth:** A site x of type i gives birth to **Poisson($\beta(i)$)** individuals, spreads them randomly on a neighborhood $\mathcal{N}_N(x)$ and then dies. The type of x = the type of a unif. choice over the individuals x received.



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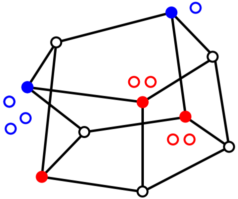

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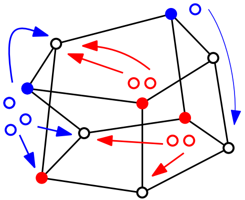
- **Initial configuration:** η_0.

Evolution: Two consecutive steps (per unit of time).

- **Growth:** A site x of type i gives birth to **Poisson(**$\beta(i)$**)** individuals, spreads them randomly on a neighborhood $\mathcal{N}_N(x)$ and then dies. The type of x = the type of a unif. choice over the individuals x received.



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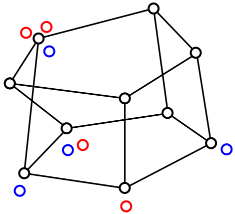

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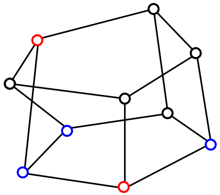

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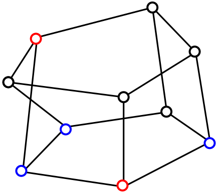
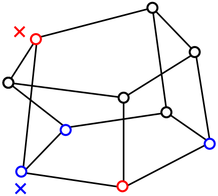


- **Epidemic:** Each site x of type i is infected with probability $\alpha_N(i)$. In this case, the infection wipes out the entire connected component of x with the same type as x.



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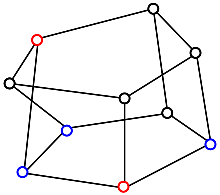

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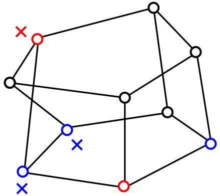

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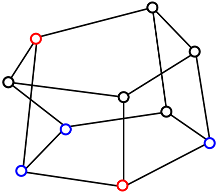

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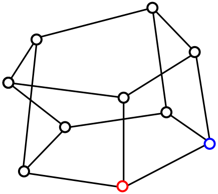

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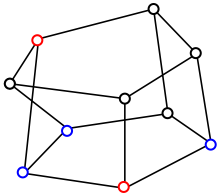

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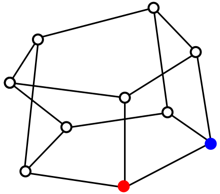

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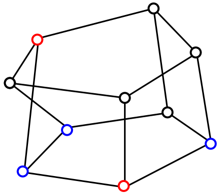
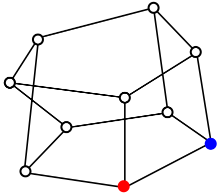


- **Epidemic:** Each site x of type i is infected with probability $\alpha_N(i)$. In this case, the infection wipes out the entire connected component of x with the same type as x.

$$\rho_k(i) = \frac{1}{N} \sum_{x \in V_N} \mathbb{1}_{\{\eta_k(x)=i\}}$$



$$\vec{\rho}_{k+1} = (\rho_{k+1}(1), \rho_{k+1}(2)) = (1/10, 1/10)$$

Theorem (Durrett & Remenik '09)

Hypothesis:

- *Number of species:* ***one***.
- *Space:* G_N ***uniform random 3-regular graphs with*** N ***vertices.***
- *Offspring:* $\beta \in (0, \infty)$ *and* $\mathcal{N}_N(x) = G_N$, *for all* $x \in V$.
- *Infection:* $\alpha_N \log(N) \to \infty$ *and* $\alpha_N \to 0$. ***(macroscopic killings).***
- *Initial density:* $\rho_0^N \xrightarrow{(d)} p$.

Then, $$(\rho_k^N)_{k\in\mathbb{N}} \xrightarrow{(d)} (h^{\circ k}(p))_{k\in\mathbb{N}}$$ *on compact time intervals.*

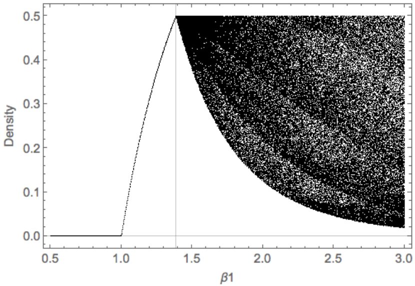


Figure: Bifurcation diagram one species $\alpha = 0$.

$$h(p) = \begin{cases} 1 - e^{-\beta p} & \text{if } 0 \le p \le a_0, \\ \frac{e^{-3\beta p}}{(1-e^{-\beta p})^2} & \text{if } a_0 < p \le 1. \end{cases}$$

a_0 explicit.

Theorem (F., Linker & Remenik '18)

Hypothesis:

- *Number of species:* ***two***.
- *Space:* G_N ***uniform random 3-regular graphs with*** N ***vertices.***
- *Offspring:* $\beta(i) \in (0, \infty)$ *and* $\mathcal{N}_N(x) = G_N$*, for all* $x \in V$.
- *Infection:* $\alpha_N(i)\log(N) \to \infty$ *and* $\alpha_N(i) \to \alpha(i) \in [0,1]$. ***(microscopic killings too).***
- *Initial density:* $\vec{\rho}_0^N \xrightarrow{(d)} \vec{p}$.

Then, $$(\vec{\rho}_k^N)_{k\in\mathbb{N}} \xrightarrow{(d)} (\vec{h}_{\vec{\alpha}}^{\circ k}(\vec{p}))_{k\in\mathbb{N}}$$ *on compact time intervals.*

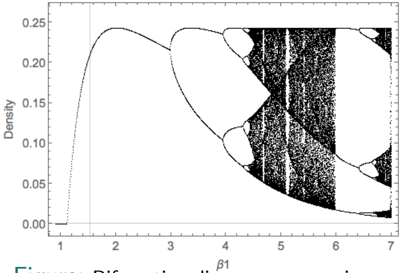


Figure: Bifurcation diagram one species $\alpha = 0.1$.

$$h_\alpha(p) = \frac{\left(1 - \sqrt{1 - 4(1-\alpha)(1-e^{-\beta p})e^{-\beta p}}\right)^3}{8(1-\alpha)^2(1-e^{-\beta p})^2}$$

Theorem (F., Linker & Remenik '18)

Hypothesis:

- *Number of species:* ***two***.
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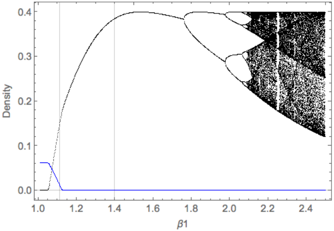
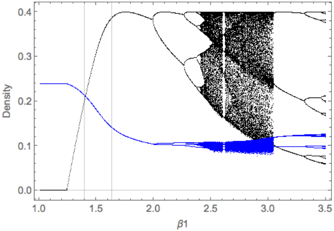


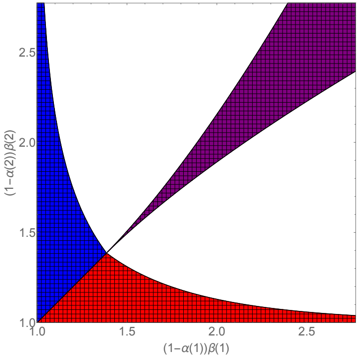
(a) BD Multi-type $\beta(1) = 1.99 log(2)$ and $\vec{\alpha} = (0.01, 0.2)$.



(b) B.D. Mult-type $\beta(1) = 2.6\log(2)$ and $\vec{\alpha} = (0.01, 0.1)$.

Theorem (F, Linker & Remenik '18)

There are explicit regions of the parameter space where the dynamical system shows: ***domination*** *(red and blue) or* ***coexistence*** *(purple).*



Theorem (F, Linker & Remenik '18)

In these regions, the stochastic system behaves as the dynamical system:

- *When there is* ***domination****, the weaker type dies out in time $O(\log(N))$.*
- *When there is* ***coexistence****, both types survive for at least $e^{\sqrt{\log(N)}}$.*

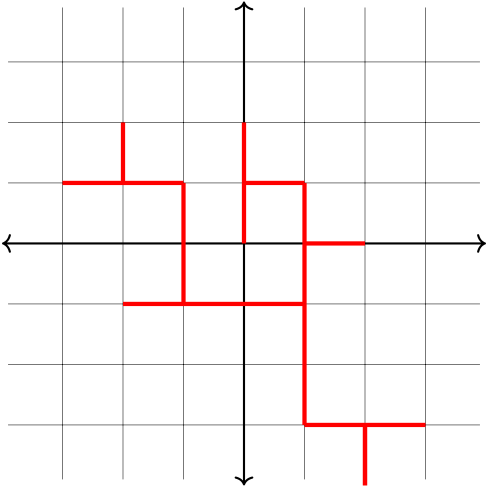


Figure: Uniform random tree of size 20 containing the origin on $\mathbb{Z}^2$.

Tree-decorated planar maps: combinatorial results.

with A. Sepúlveda.

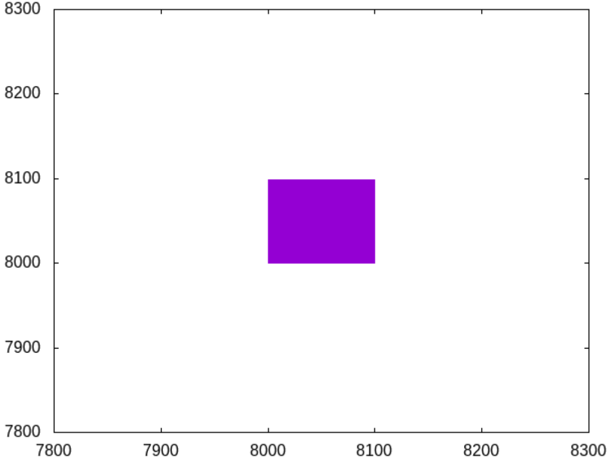
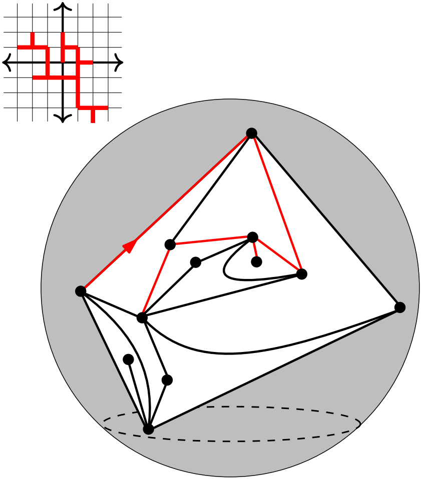
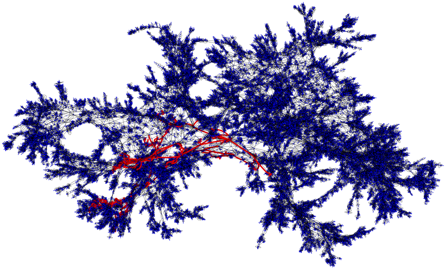


Figure: Dynamic on trees of size 10000.

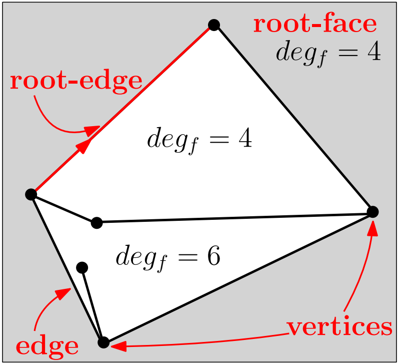


(a) tree-decorated quad. 10 faces, tree of size 6.



(b) Unif. tree-decorated quad. 90k faces and tree of size 500.

- A **face**= A connected component of the complement of the edges.
- The **root-edge**= distinguished half edge.
- The **root-face**= face to the left of the root-edge.
- **Degree of a face**= number of adjacent edges to it.



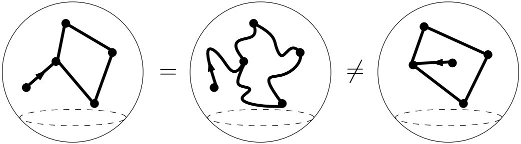
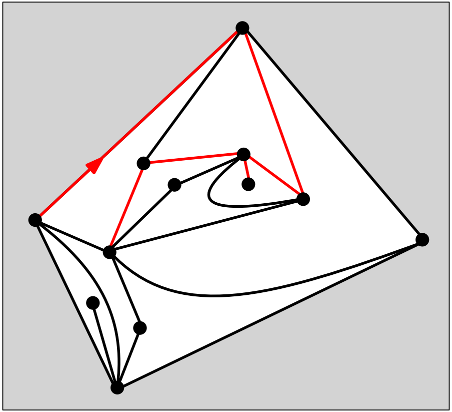


Figure: Same graph, different embeddings on the sphere.

A (f, a) **tree-decorated map** is a pair $(\mathfrak{m}, \mathfrak{t})$ where:

- $\mathfrak{m}$ is a rooted map with f faces.
- $\mathfrak{t}$ is a submap of $\mathfrak{m}$ ($\mathfrak{t} \subset_M \mathfrak{m}$).
- $\mathfrak{t}$ is a tree with a edges.
- $\mathfrak{t}$ contains **the root-edge of $\mathfrak{m}$**.



~~Spanning~~ tree-decorated maps

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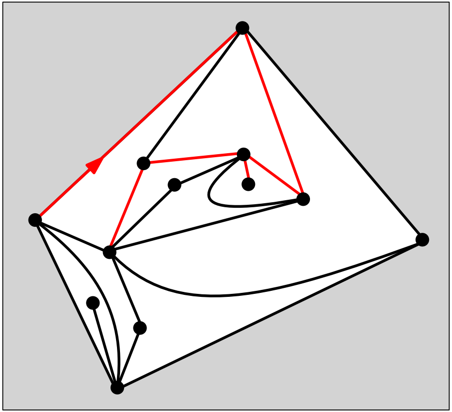


It interpolates: In the case of quadrangulations

- $\underline{a = 1 \rightarrow}$ quadrangulations with f faces. **[Tutte '60, Bender & Canfield '94, Cori-Vauquelin-Schaeffer '98, Schaeffer '97, Bettinelli '15]**
- $\underline{a = f + 1 \rightarrow}$ spanning-tree decorated quadrangulations with f faces. **[Mullin '67, Walsh & Lehman '72, Cori et al '86; Bernardi '06]**

Counting results

Theorem (F. & Sepúlveda '19)

The number of (f, a) tree-decorated quadrangulations is

$$3^{f-a} \frac{(2f + a - 1)!}{(f + 2a)!(f - a + 1)!} \frac{2a}{a + 1} \binom{3a}{a, a, a}$$

We also count

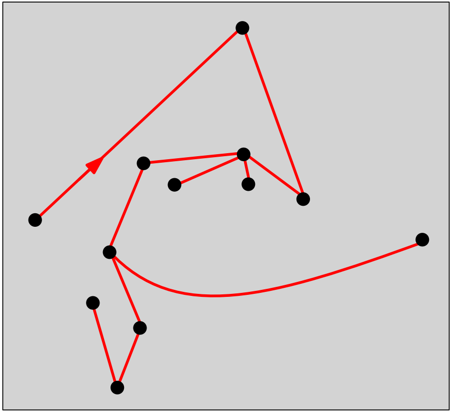
- (f, a) tree-decorated triangulations.
- Maps (triangulations and quadrangulations) with a simple boundary decorated in a subtree.
- Forest-decorated maps.
- "Tree-decorated general maps".

Planar trees

A **planar tree** is a rooted map with one face.

Number of planar trees with a edges

$$\mathcal{C}_a = \frac{1}{a+1}\binom{2a}{a}.$$



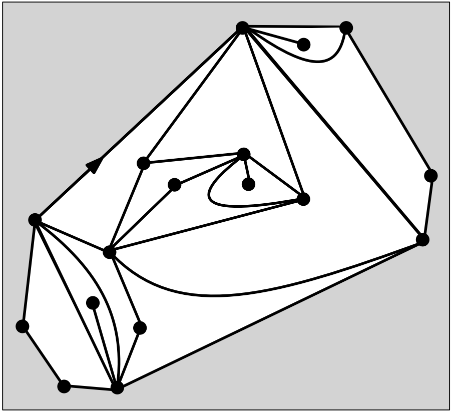
Quadrangulations with a **simple** boundary

Number of rooted-quadrangulations with:

- f internal faces.
- **simple boundary** of size $2p$ (root-face of degre $2p$).

$$\frac{3^{f-p}2p}{(f+2p)(f+2p-1)}\binom{2f+p-1}{f-p+1}\binom{3p}{p}.$$

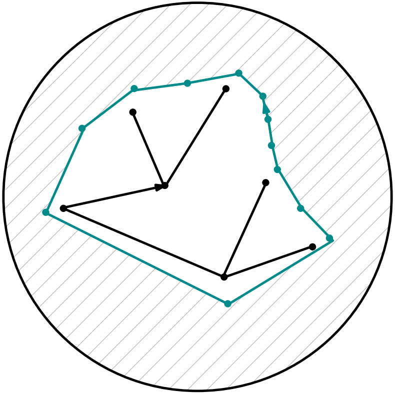
Analytic **[Bouttier & Guitter '09]** and bijective **[Bernardi & Fusy '17]**.



Bijection

Theorem (F. & Sepúlveda '19)

The set of (f, a) tree-decorated maps is in bijection with
(the set of maps with a simple boundary of size $2a$ and f interior faces)
$\times$ (the set of trees with a edges).



Remarks and extensions

The bijection makes a correspondence between:

[Tree-decorated map]		**[Map with a boundary, Tree]**
Faces of degree q	$\longleftrightarrow$	Internal faces of degree q
Internal vertices of degree d	$\longleftrightarrow$	Internal vertices of degree d
Internal edges	$\longleftrightarrow$	Internal edges
Corner of the tree	$\longleftrightarrow$	Boundary vertices.

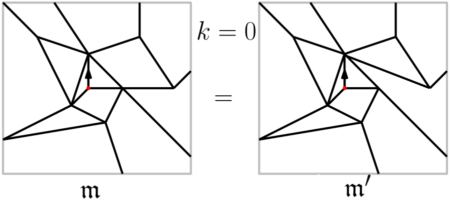
- We can restrict the bijection to q-angulations.
- It can be restricted to some subfamilies of trees:
  1. Binary tree- decorated Maps.
  2. SAW decorated maps (Already done by Caraceni & Curien).

Notation: $\mathfrak{q}_f^a$= Unif. tree-decorated quad. with f faces and a tree of size a.
Consider:

- $B_r(\mathfrak{m})$ = ball of radius r from the root-vertex.
- $\mathcal{M}$= set of (locally finite) maps.

We endowed $\mathcal{M}$ with the (local) topology induced by

$$\mathrm{d_{loc}}(\mathfrak{m}_1, \mathfrak{m}_2) = (1 + \sup\{\mathrm{r} \geq 0 : \mathrm{B_r}(\mathfrak{m}_1) = \mathrm{B_r}(\mathfrak{m}_2)\})^{-1}$$



Proposition

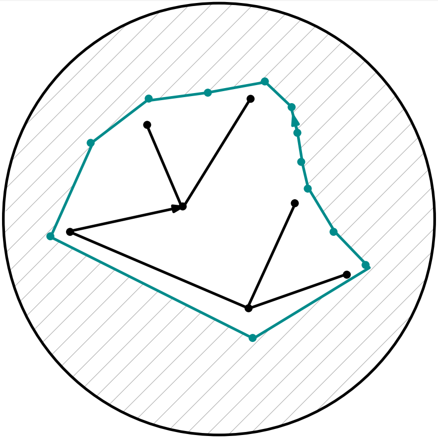
The space $(\overline{\mathcal{M}}, \mathrm{d_{loc}})$ is Polish (metric, separable and complete).

Result

Notation: $\mathfrak{q}_f^a$= Unif. tree-decorated quad. with f faces and a tree of size a.

Theorem (F. & Sepúlveda '19+)

$$\mathfrak{q}_f^a \xrightarrow[local, f\to\infty]{(d)} \mathfrak{q}_\infty^a \xrightarrow[local, a\to\infty]{(d)} \mathfrak{q}_\infty^\infty$$



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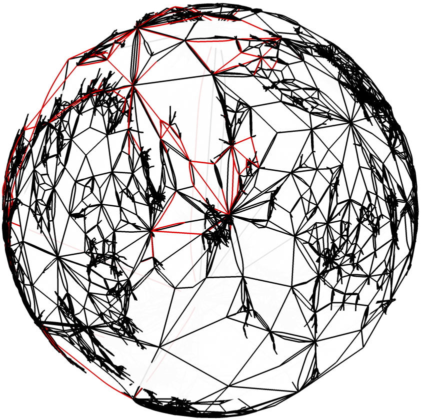
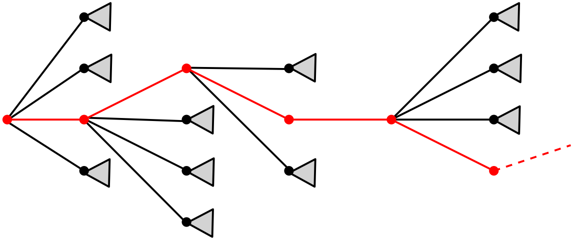


$\mathfrak{t}_a$= Unif. tree with a edges.

Theorem (Kesten '86)

$$\mathfrak{t}_a \xrightarrow[local]{(d)} \mathfrak{t}_\infty$$



$\mathfrak{t}_\infty$ **construction.**

Uniform quadrangulation with a boundary

$\mathfrak{q}^S_{f,p}$= Unif. quadrangulations with a simple boundary of size $2p$ and f faces.

Theorem (Curien & Miermont '12)

$$\mathfrak{q}^S_{f,p} \xrightarrow[local(f\to\infty)]{(d)} \mathfrak{q}^S_{\infty,p} \xrightarrow[local(p\to\infty)]{(d)} UIHPQ^S$$

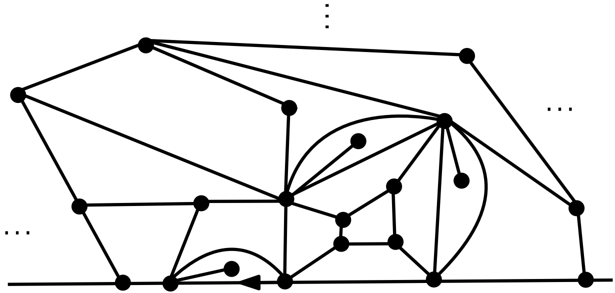


Figure: sketch of a $UIHPQ^S$.

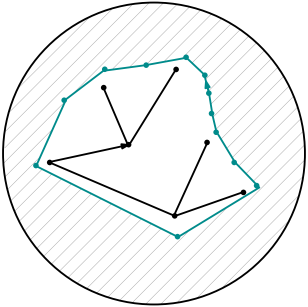
Local limit results

Notation: $\mathfrak{q}_f^a$= Unif. tree-decorated quad. with f faces and a tree of size a.

Theorem (F. & Sepúlveda '19+)

$$\mathfrak{q}_f^a \xrightarrow[local, f\to\infty]{(d)} \mathfrak{q}_\infty^a \xrightarrow[local, a\to\infty]{(d)} \mathfrak{q}_\infty^\infty$$

$\mathfrak{q}_\infty^\infty$ is the "gluing" of $\mathfrak{t}_\infty$ and $UIHPQ^S$.



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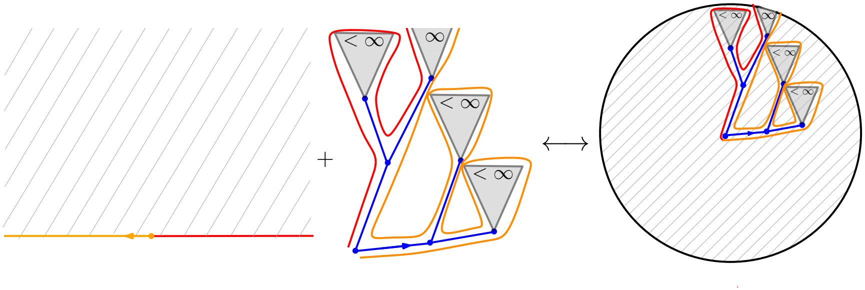
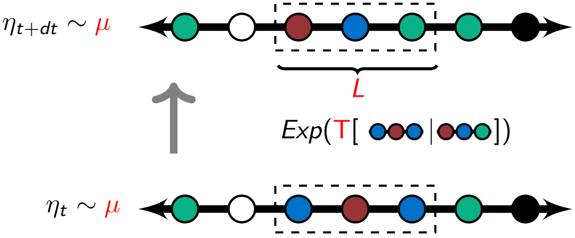


Definition

Given T, a distribution μ is said to be *invariant* if $\eta_t \sim \mu$ for any $t \geq 0$, when $\eta_0 \sim \mu$.



<u>Well definition of PS:</u>

Can we define a Markov process with jumps according to T**?**

When well defined, there is a correspondence between G_T Markovian generator and $\{P_t\}_{t\geq 0}$ Markovian semigroup; and the following are satisfied

$$\mu_t f = \mu_0 P_t f$$
$$\frac{\partial \mu_t f}{\partial t} = \int G_\mathrm{T} f d\mu_t$$

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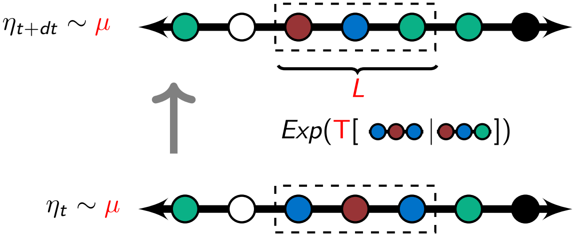

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When well defined, there is a correspondence between G_{T} Markovian generator and $\{P_t\}_{t\geq 0}$ Markovian semigroup; and the following are satisfied

$$\mu f = \mu P_t f$$

$$\frac{\partial \mu_t f}{\partial t} = \int G_{\mathrm{T}} f d\mu = 0$$

Application

Full characterization of (M, T) such that M is invariant by T. Case **2 colors ($\kappa = 2$), memory 1 ($m = 1$) and range 2 ($L = 2$)**.

```
Cycle7MT:=proc(a,b,c,d) option remember: local Vec:
    Vec :=[ a, b, c, d, 0, 0, 0]:
    return( simplify( add(Z4( [seq( Vec[modu(i +j, 7)],j=1..4) ] ),i=1..7))):
end:
>

for i in {0,1} do M[i,0]:=1-M[i,1]: od:

>
SystemEquation := {  seq( Cycle7MT(op(u)), u in ensem(4,K)), seq( M[op(u)]*g[op(u)]-1,u in ensem(2,K)), (M[0,1]-M[1,1])*x-1}:
#########################
PolynomialSystem:= map(x-> numer(normal(x)), SystemEquation): # numerators only
Variables:= seq(g[op(u)],u in ensem(2,K)),x, seq(seq(M[i,j],i=0..K-1),j=1..K-1) , NonZeroT :
#########################
# grobner
G:= Basis(PolynomialSystem,plex(  Variables)):
Affiche(G,2000);
unassign('M'); # for further computations
```

$$K := 2$$

"Parameters = NonZeroTransition", $t_{0,1}, t_{0,2}, t_{0,3}, t_{1,0}, t_{1,2}, t_{1,3}, t_{2,0}, t_{2,1}, t_{2,3}, t_{3,0}, t_{3,1}, t_{3,2}$

1

$$t_{3,0}$$

2

$$t_{2,1}$$

3

$$t_{1,2}$$

4

$$t_{0,3}$$

5

$$t_{0,1}t_{1,0} + t_{0,1}t_{2,0} + t_{0,2}t_{1,0} + t_{0,2}t_{2,0} - t_{1,3}t_{3,1} - t_{1,3}t_{3,2} - t_{2,3}t_{3,1} - t_{2,3}t_{3,2}$$

6

$$M_{1,1}^2 t_{1,0} + M_{1,1}^2 t_{2,0} - M_{1,1}^2 t_{3,1} - M_{1,1}^2 t_{3,2} - M_{1,1} t_{1,0} - M_{1,1} t_{1,3} - M_{1,1} t_{2,0} - M_{1,1} t_{2,3} + t_{1,3} + t_{2,3}$$

Grobner's basis

Generalization of Gaussian elimination for linear systems.

• For a set of polynomials $\mathcal{P}$, the Grobner basis finds a "minimal representation" of the ideal generated by this set in the ring of polynomials with coefficients in a field (here $\mathcal{C}$).

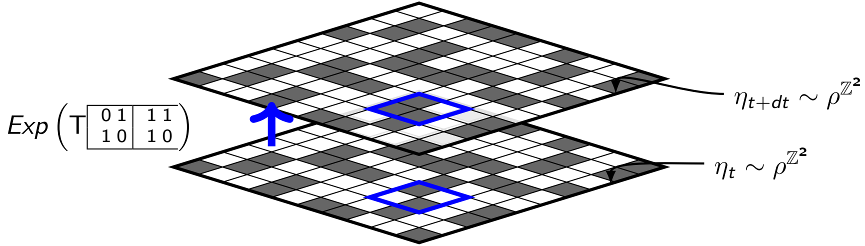
•"Minimal representation" = "small" generator of the ideal.

•"Small"= with respect to a certain order of monomials. Basically a way to do the sequence of divisions of polynomials (which is a generalized version of Gaussian division).

•If the result of a Grobner basis gives $\{1\}$, it means that there is no solution (in a sense 1=0 generates the ideal).

• If the result is different that the constant, then it assures the existence of solutions in the field $\mathcal{C}$.

Multi-dimensional case: invariance of product measures



Consider the three following sets:

$$\Gamma_0 = \{(0,0),(0,1),(1,0)\}, \quad \Gamma_1 = \Gamma_0 \cup \{(2,0)\}, \quad \Gamma_2 = \Gamma_1 \cup \{(1,1)\}.$$

Theorem 2D

Let $\kappa < +\infty$. Consider ρ a probability distribution with full support on E_κ and $\mathrm{T} = \left[\mathcal{T}_u v\right]_{u,v \in E_\kappa^{\mathbf{Sq}}}$ a JRM indexed by 2x2 squares. The measure $\rho^{\mathbb{Z}^2}$ is invariant by T on $\mathbb{Z}^2$ iff the two following conditions hold simultaneously:

- $\mathrm{NLine}^{\rho,\mathrm{T}} \equiv 0$ on $E_\kappa^{\Gamma_0}$,
- for any $x \in E_\kappa^{\Gamma_2}$,

$$\mathrm{NLine}^{\rho,\mathrm{T}}(x) - \mathrm{NLine}^{\rho,\mathrm{T}}(x(\Gamma_1)) = 0. \qquad (1)$$

Scaling limit conjecture

Conjecture (F. & Sepúlveda '19+)

Let $(\mathfrak{m}, \mathfrak{t})$ be a Unif. tree-decorated map with f faces and boundary of size $a(f)$ with $a(f) = O(f^{\alpha})$. Depending on α as $f \to \infty$

$$\left((\mathfrak{m}, \mathfrak{t}), \frac{\mathrm{d}_{\mathrm{map}}}{f^{\beta}}\right) \xrightarrow[GH]{(d)} \begin{cases} \textit{Brownian map} & \text{if } \alpha < 1/2, \beta = 1/4 \textit{(Proved)} \\ \textbf{\textit{Shocked map}} & \text{if } \alpha = 1/2, \beta = 1/4 \textit{(In progress)} \\ \textit{Tree-decorated map} & \text{if } \alpha > 1/2, \\ & \qquad \beta = \left(2\chi - \frac{1}{2}\right)\alpha - \chi + \frac{1}{2} \end{cases}$$

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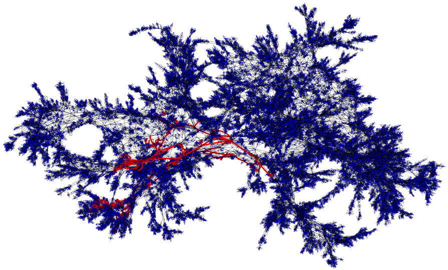

Shocked map

Shocked map properties:

- **It is not degenerated** (Proved).
- It should be the gluing of a Brownian disk and a CRT.
- Hausdorff dim. 4 (Proved).
- The tree has Hausdorff dim. 2 (In progress, ≤ 2 proved).
- Homeomorphic to $\mathbb{S}^2$. (Proved).

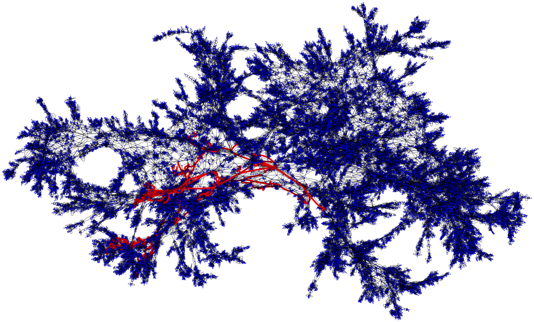

Figure: Unif. (90k,500) tree-decorated quadrangulation.

Brownian Disk

$\mathfrak{q}_{f,p}$= Unif. quadrangulations with boundary $2p$ and f faces.
For a sequence $(p(f))_{f\in\mathbb{N}}$, define $\bar{p} = \lim p(f)f^{-1/2}$ as $f \to \infty$.

Theorem (Scaling limit (Bettinelli '15))

$$\left(\mathfrak{q}_{f,p(f)}, \frac{\mathrm{d}_{\mathrm{map}}}{s(f,p(f))}\right) \xrightarrow[GH]{(d)} \begin{cases} \textit{Brownian map} & \textit{if } s(f,p(f)) = f^{1/4} \textit{ and } \bar{p} = 0 \\ \textbf{\textit{Brownian disk}} & \textit{if } s(f,p(f)) = f^{1/4} \textit{ and } \bar{p} \in (0,+\infty) \\ \textit{CRT} & \textit{if } s(f,p(f)) = 2p(f)^{1/2} \textit{ and } \bar{p} = \infty \end{cases}$$

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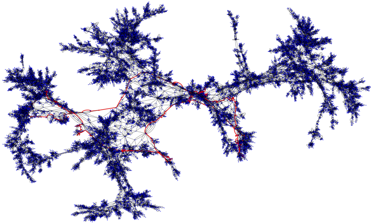
Properties (Bettinelli & Miermont '15)

Brownian disk properties

- *The boundary is simple.*
- *Hausdorff dim. 4 in the interior, 2 in the boundary.*
- *Homeomorphic to the disk 2d.*



Unif. quad. with 30k interior faces and boundary 173.

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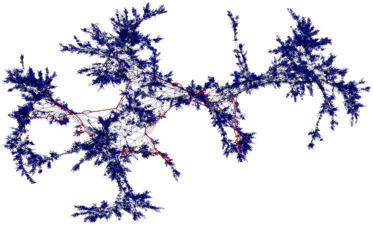

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