Bijections for tree-decorated map and applications to random maps.

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(Work in progress with Avelio Sepúlveda (Univ. Lyon 1))

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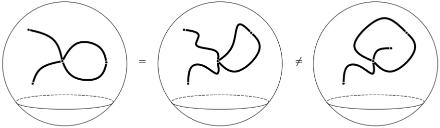
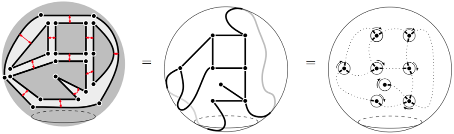
MAPS

Map



A **planar map** is a proper embedding of a finite connected planar graph in the sphere, considered up to direct homeomorphisms of the sphere.

Same graph, different embeddings on the sphere (sketch by N. Curien)

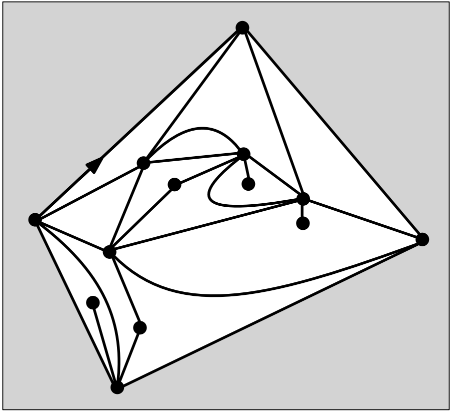


Maps seen as different objects (sketch by N. Curien)

Map

The **faces** are the connected components of the complement of the edges. It has a distinguished half-edge: the **root edge**.

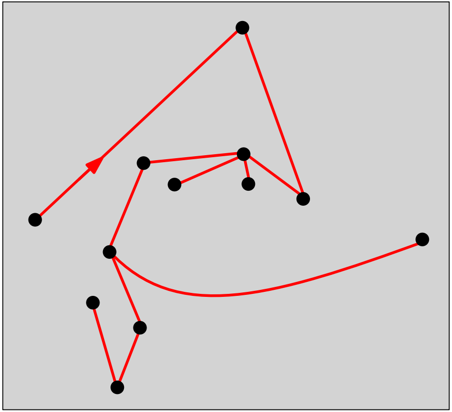
The face that is at the left of the root-edge will be called the **root-face**.



Planar trees

A **planar tree** is a map with one face.
The set of trees with a edges.

$$\mathcal{C}_a = \frac{1}{a+1}\binom{2a}{a}$$



Quadrangulations

The **degree of a face** is the number of edges adjacent to it.

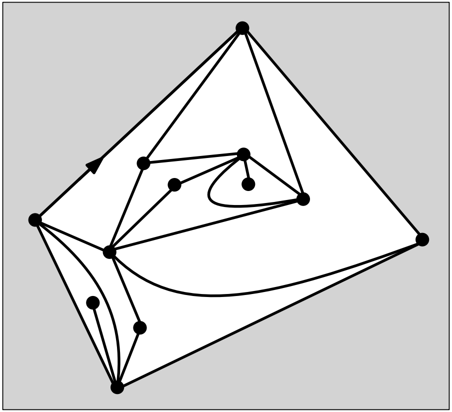
A **quadrangulation** is a map whose faces have degree 4. Let $\mathcal{Q}_f$ be the set of all quadrangulations with f faces, then

$$|\mathcal{Q}_f| = 3^f \frac{2}{f+2} \underbrace{\frac{1}{f+1}\binom{2f}{f}}_{\mathcal{C}_f}.$$

Analytic [Tutte '60].

This number also counts general maps with $a = f$ edges!

Bijective [Tutte '60].



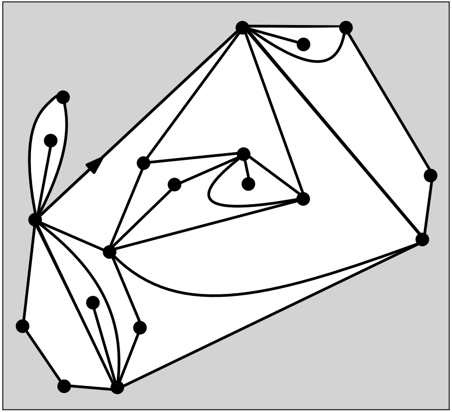
Quadrangulations with a boundary

A quadrangulation with a boundary is a map where the **root-face** plays a special role: it has **arbitrary degree**.

The set of quadrangulations with f internal faces and a boundary of size $2p$ has cardinality

$$\frac{3^f p}{(f+p+1)(f+p)} \binom{2f+p-1}{f} \binom{2p}{p}.$$

Analytic by [Bender & Canfield '94; Bouttier & Guitter '09] and bijective by [Schaeffer '97 ; Bettinelli '15]

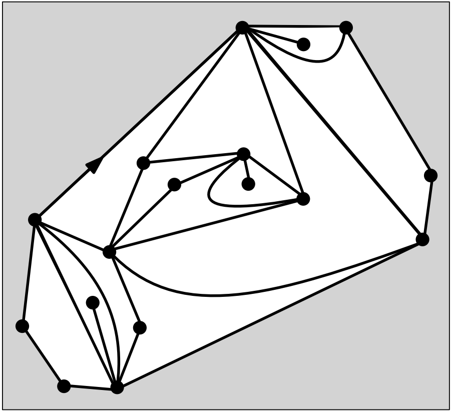


Quadrangulations with a **simple** boundary

The set of quadrangulations with f internal faces and a **simple boundary** of size p (root-face of degre p) has cardinality

$$\frac{3^{f-p}2p}{(f+2p)(f+2p-1)}\binom{2f+p-1}{f-p+1}\binom{3p}{p}.$$

Analytic [Bouttier & Guitter '09]



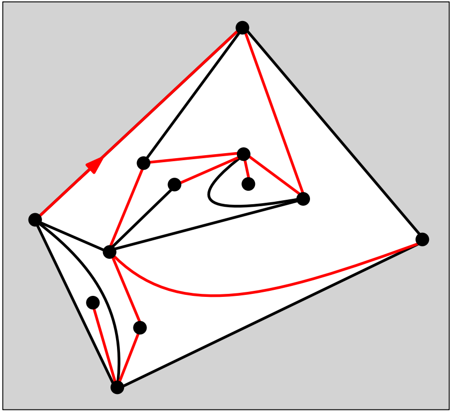
Spanning tree-decorated maps

A spanning tree-decorated map (**ST map**) is a pair $(\mathfrak{m}, \mathfrak{t})$ where $\mathfrak{m}$ is a map and $\mathfrak{t} \subset_M \mathfrak{m}$ is a spanning tree of $\mathfrak{m}$.

The family of ST maps with a edges is counted by

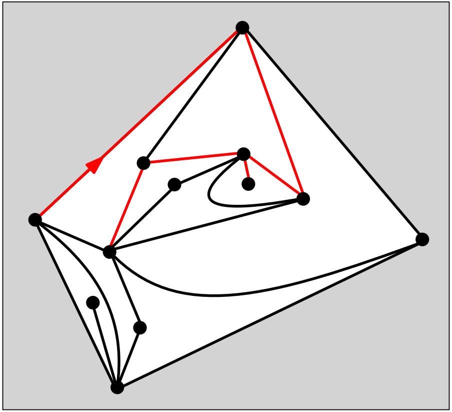
$$\mathcal{C}_a \mathcal{C}_{a+1}$$

Analytic by [Mullin '67] and bijective by [Walsh and Lehman '72; Cori, Dulucq & Viennot '86; Bernardi '06]



~~Spanning~~ tree-decorated maps

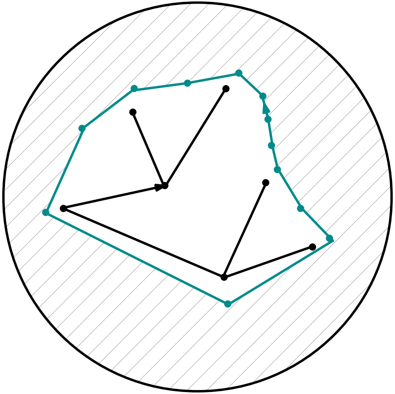
A (f, a) **tree-decorated map** is a pair $(\mathfrak{m}, \mathfrak{t})$ where $\mathfrak{m}$ is a map with f faces, and $\mathfrak{t}$ is a tree with a edges, so that $\mathfrak{t} \subset_M \mathfrak{m}$ **containing the root-edge**.



Bijection

Proposition (F. & Sepúlveda '19)

The set of (f, a) tree-decorated maps is in bijection with (the set of maps with a simple boundary of size 2a and f interior faces) $\times$ (the set of trees with a edges).



What do we obtain when the boundary is not simple?

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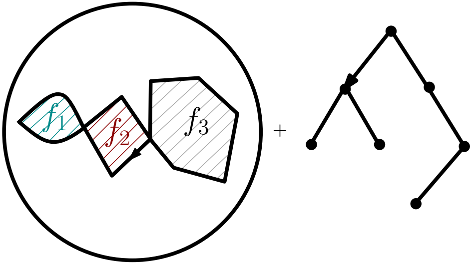

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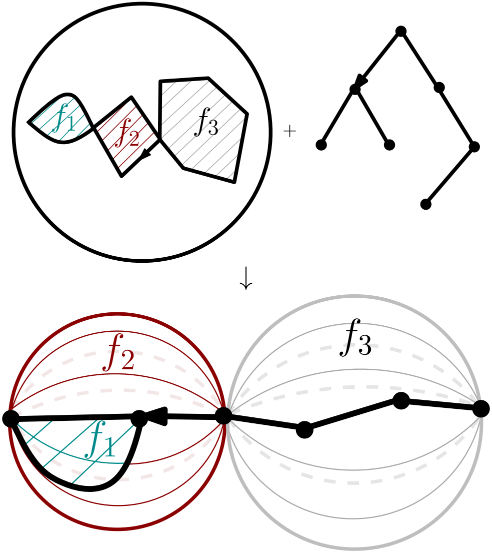

Counting results

Corollary (F. & Sepúlveda '19)

The number of (f, a) tree-decorated quadrangulations is

$$3^{f-a}\frac{(2f+a-1)!}{(f+2a)!(f-a+1)!}\frac{2a}{a+1}\binom{3a}{a,a,a}$$

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We also count

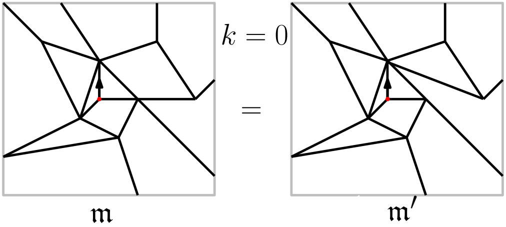
- (f, a) tree-decorated triangulations.
- Maps (triangulations and quadrangulations) with a simple boundary decorated in a subtree.
- Forest-decorated maps.
- "Tree-decorated general maps".
- Maps decorated in different subfamilies of trees.

CONVERGENCE RESULTS

Local Limits (Benjamini-Schramm Topology '01)

For a map $\mathfrak{m}$ and $r \in \mathbb{N}$, let $B_r(\mathfrak{m})$ denote the ball of radius r from the root-vertex. Consider $\mathcal{M}$ a family of finite maps. The **local topology** on $\mathcal{M}$ is the metric space $(\mathcal{M}, \mathsf{d}_{\mathsf{loc}})$, where

$$\mathsf{d}_{\mathsf{loc}}(\mathfrak{m}_1, \mathfrak{m}_2) = (1 + \sup\{\mathsf{r} \geq 0 : \mathsf{B}_\mathsf{r}(\mathfrak{m}_1) = \mathsf{B}_\mathsf{r}(\mathfrak{m}_2)\})^{-1}$$



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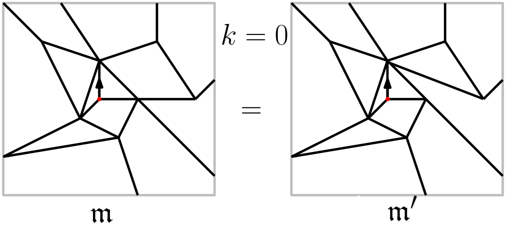

Proposition

The space $(\overline{\mathcal{M}}, \mathsf{d}_{\mathsf{loc}})$ is Polish (metric, separable and complete).

Gromov-Hausdorff topology

Let (E, d_E) be a metric space and $A, B \subset E$. The **Hausdorff distance** is

$$\mathrm{d_H(A,B) = \max\left\{\sup_{x\in B} d_E(x,A), \sup_{y\in A} d_E(y,B)\right\}}$$

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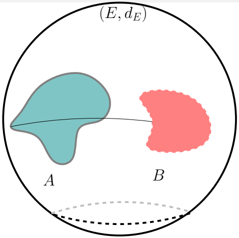

Gromov-Hausdorff topology

Consider the set S of compact metric spaces up to isometry classes. The **Gromov-Hausdorff distance** between two metric spaces (X, d) and (X', d') is defined as

$$\mathrm{d_{GH}}((\mathrm{X}, \mathrm{d}), (\mathrm{X}', \mathrm{d}')) = \inf \mathrm{d_H}(\phi(\mathrm{X}), \phi'(\mathrm{X}'))$$

where the infimum is taken over all metric spaces $(E, \mathrm{d_E})$ and all isometric embeddings ϕ, ϕ' from X, X' respectively into E.

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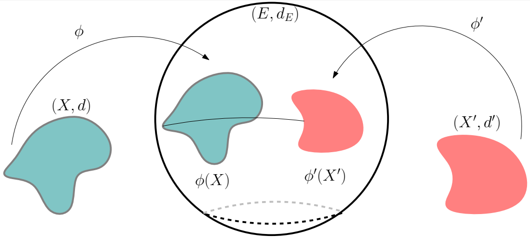

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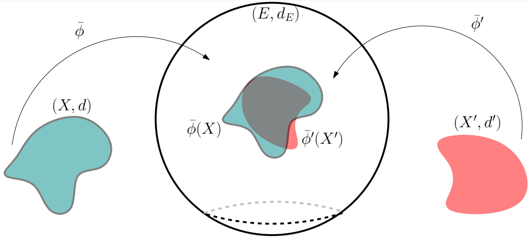

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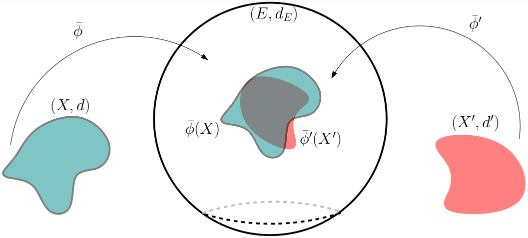

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where the infimum is taken over all metric spaces $(E, \mathrm{d_E})$ and all isometric embeddings ϕ, ϕ' from X, X' respectively into E.

Proposition

The function $\mathrm{d_{GH}}$ *induces a metric on* S*. The space* $(S, \mathrm{d_{GH}})$ *is separable and complete.*

Uniform Trees

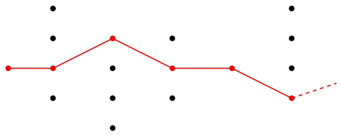
t_a= Unif. tree with a edges.

Theorem (Kesten '86)

$$t_a \xrightarrow[local]{(d)} t_\infty$$

Properties

- *t_∞ is an infinite tree.*
- *It has one infinite branch (the spine) which divides the tree in independent critical geometric Galton-Watson trees.*



t_∞ **construction.**

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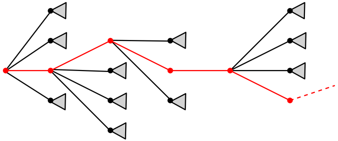

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Theorem (Aldous '91)

$$\left(\mathfrak{t}_a, \frac{d_{\text{Tree}}}{a^{1/2}}\right) \xrightarrow[GH]{(d)} CRT$$

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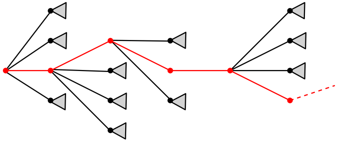
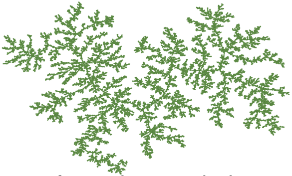
Properties

- *The CRT is a tree.*
- *Almost every point is a leaf.*
- *Hausdorff dimension 2.(Duquesne & Le Gall '05)*



$\mathfrak{t}_\infty$ construction.



Uniform random tree 50k edges.

Uniform quadrangulations

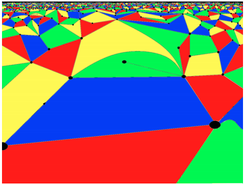
$\mathfrak{q}_f$= Unif. quadrangulation with f faces.

Theorem (Krikun '06)

$$\mathfrak{q}_f \xrightarrow[local]{(d)} UIPQ$$

Properties

- *The UIPQ is an infinite quad.*
- *The vol. and per. of the exploration on it have been studied (Curien & Le Gall '14).*



(Sketch by N. Curien)

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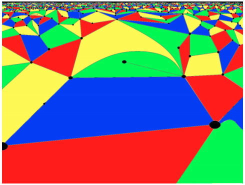
Theorem (Miermont '13, Le Gall '13)

$$\left(\mathfrak{q}_f, \frac{\mathrm{d}_{\mathrm{map}}}{f^{1/4}}\right) \xrightarrow[GH]{(d)} \textit{Brownian map}$$

Properties

- *Hausdorff dim. is 4 (Le Gall '07).*
- *Homeomorphic to $\mathbb{S}^2$ (Le Gall & Paulin '08).*



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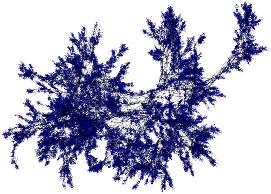

Unif. quadrangulation 30k faces.

Uniform quadrangulation with a boundary

$\mathfrak{q}_f^p$= Unif. quadrangulations with a boundary of size $2p$ and f faces.

Theorem (Curien & Miermont '12)

$$\mathfrak{q}_f^p \xrightarrow[local(f\to\infty)]{(d)} \mathfrak{q}_\infty^p \xrightarrow[local(p\to\infty)]{(d)} UIHPQ$$

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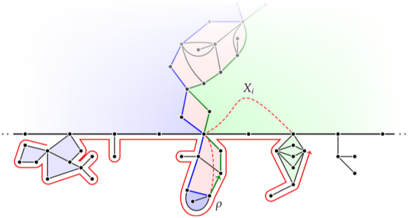
Properties (Curien & Miermont '12)

- $\mathfrak{q}_\infty^p$ = *Uniform Infinite Planar Quadrangulation with perimeter p.*
- *They also obtain the convergences for the simple boundary case.*
- *The* $\mathfrak{q}_\infty^p$ *has one infinite component, called the core. Moreover,*

$$\frac{\partial Core(\mathfrak{q}_\infty^p)}{2p} \xrightarrow[p\to\infty]{(prob)} \frac{1}{3}.$$



UIHPQ (sketch by N. Curien & A. Caraceni).

Brownian Disk

$\mathfrak{q}_f^p$= Unif. quadrangulations with boundary $2p$ and f faces.
For a sequence $(p(f))_{f\in\mathbb{N}}$, define $\bar{p} = \lim p(f) f^{-1/2}$ as $f \to \infty$.

Theorem (Scaling limit (Bettinelli '15))

$$\left(\mathfrak{q}_f^{p(f)}, \frac{\mathrm{d}_{\mathrm{map}}}{s(f,p(f))}\right) \xrightarrow[GH]{(d)} \begin{cases} \textit{Brownian map} & \textit{if } s(f,p(f)) = f^{1/4} \textit{ and } \bar{p} = 0 \\ \textbf{\textit{Brownian disk}} & \textit{if } s(f,p(f)) = f^{1/4} \textit{ and } \bar{p} \in (0,+\infty) \\ \textit{CRT} & \textit{if } s(f,p(f)) = 2p(f)^{1/2} \textit{ and } \bar{p} = \infty \end{cases}$$

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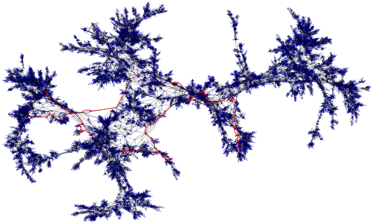
Properties (Bettinelli & Miermont '15)

Brownian disk properties

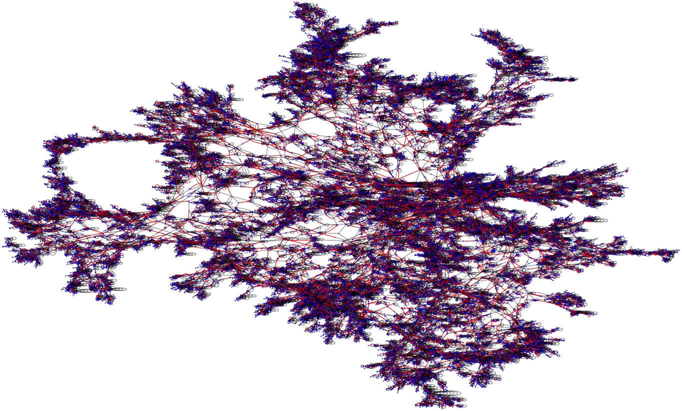
- *The boundary is simple.*
- *Hausdorff dim. 4 in the interior, 2 in the boundary.*
- *Homeomorphic to the disk 2d.*



Unif. quad. with 30k interior faces and boundary 173.

Uniform ST map

- Expected diameter is of order n^{χ} for $0.275 \leq \chi \leq 0.288$ (Ding & Gwynne '18, Gwynne, Holden & Sun '16).
- The limit (if it exists) seems not to the Brownian map.
- Convergence for the local topology (Sheffield '11).

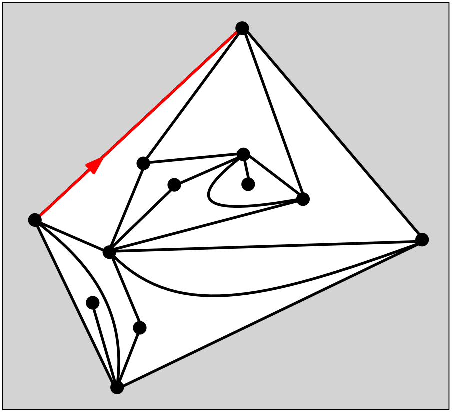


Uniform ST map 100k edges.

Uniform tree-decorated maps

$\mathfrak{q}_f^a$= Unif. tree-decorated map with f faces and a tree of size a.

Why it is interesting to study this family??



Uniform tree-decorated maps

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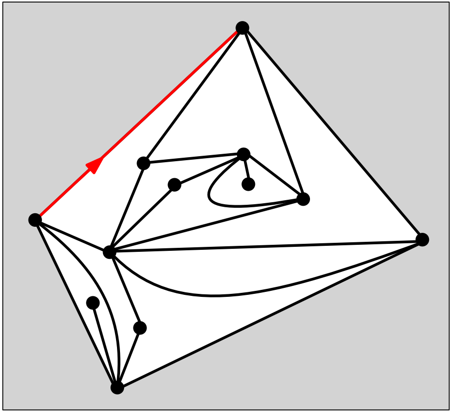
Why it is interesting to study this family??

- **New statistical mechanic family**

$\mathbb{P}(\mathfrak{q}_f^a = (\mathfrak{m}, \cdot)) \propto \#\{\text{trees of size } a \text{ in } \mathfrak{m}\}$



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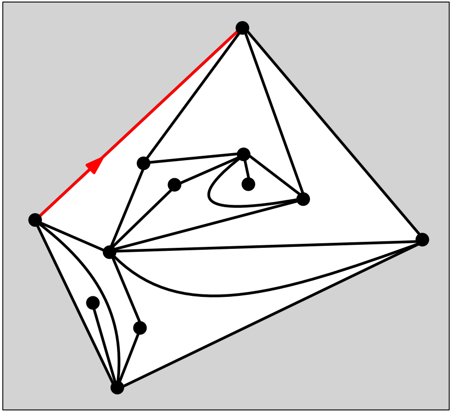
$\mathbb{P}(\mathfrak{q}_f^a = (\mathfrak{m}, \cdot)) \propto \#\{\text{trees of size } a \text{ in } \mathfrak{m}\}$

- **It interpolates**
  - $a = 1$= Uniform quadrangulations.
  - $a = f + 1$= Uniform ST quadrangulations.



Local limit results

Is there any local limit
for the **gluing of $\mathfrak{q}_{\infty}^{2a}$ with simple boundary** and **with $\mathfrak{t}_a$** as $a \to \infty$?

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Proposition (F. & Sepúlveda '19+)

There exists a local limit for this gluing and it is the gluing root to root of $\mathfrak{t}_\infty$ with a ***UIHPQ$_S$****, seeing from the root of the gluing.*

Is there any local limit
for the **gluing of $\mathfrak{q}_\infty^{2a}$ with simple boundary** and **with $\mathfrak{t}_a$** as $a \to \infty$?

Proposition (F. & Sepúlveda '19+)

*There exists a local limit for this gluing and it is the gluing root to root of $\mathfrak{t}_\infty$ with a **UIHPQ**$_S$, seeing from the root of the gluing.*

Remark

We obtain more local limits.

Scaling limit results

Corollary (F. & Sepúlveda '19+)

Let $(\mathfrak{m}, \mathfrak{t})$ *be a Unif. tree-decorated map with* f *faces and boundary of size* $a(f)$ *with* $a(f) \leq f+1$. *Then as* $a(f) \to \infty$,

$$\left(\mathfrak{t}, \frac{\mathrm{d}_{\mathsf{Tree}}}{a(f)^{1/2}}\right) \xrightarrow[GH]{(d)} CRT.$$

Scaling limit conjecture

Conjecture (F. & Sepúlveda '19+)

Let $(\mathfrak{m}, \mathfrak{t})$ *be a Unif. tree-decorated map with* f *faces and boundary of size* $a(f)$ *with* $a(f) = O(f^\alpha)$*. Depending on* α *as* $f \to \infty$

$$\left((\mathfrak{m}, \mathfrak{t}), \frac{\mathrm{d}_{\mathrm{map}}}{f^\beta}\right) \xrightarrow[GH]{(d)} \begin{cases} \textit{Brownian map} & \textit{if } \alpha < 1/2, \beta = 1/4 \textit{(Proved)} \\ \textbf{\textit{Shocked map}} & \textit{if } \alpha = 1/2, \beta = 1/4 \textit{(In progress)} \\ \textit{Tree-decorated map} & \textit{if } \alpha > 1/2, \\ & \quad \beta = \left(2\chi - \frac{1}{2}\right)\alpha - \chi + \frac{1}{2} \end{cases}$$

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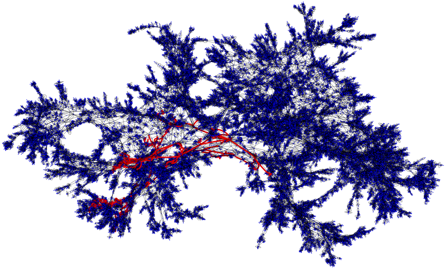


Shocked map

Shocked map properties:

- **It is not degenerated** (Proved).
- It should be the gluing of a Brownian disk and a CRT.
- Hausdorff dim. 4 (Proved).
- The tree has Hausdorff dim. 2 (In progress, ≤ 2 proved).
- Homeomorphic to $\mathbb{S}^2$. (Proved).

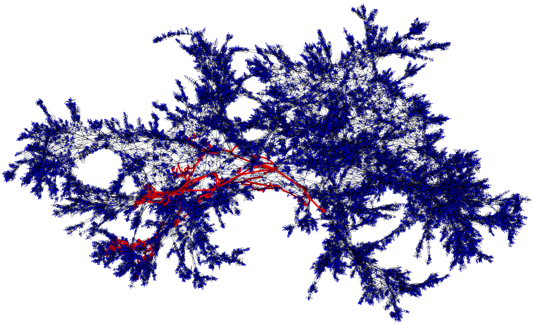


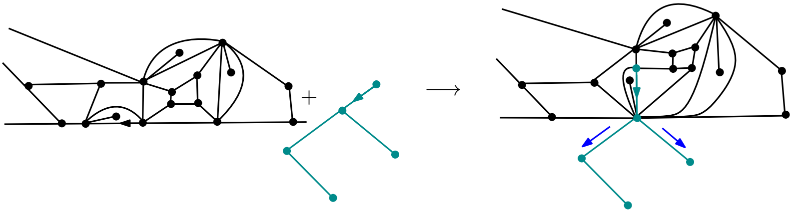
Figure: Unif. (90k,500) tree-decorated quadrangulation.

Why shocked?

Thanks for your attention!

It is not degenerated.

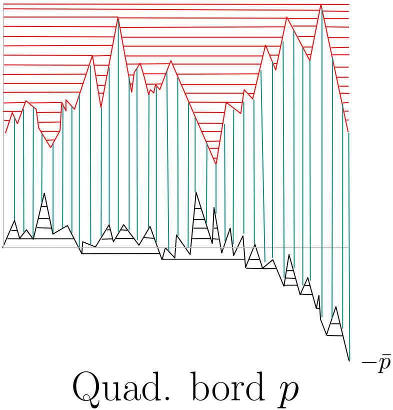
To prove it we do a sequential gluing, tool used to define a peeling.



Then we use the estimates in [Curien & Caraceni, Self-Avoiding Walks on the UIPQ] and the properties of the contour of a tree, to show that distances do not create big shortcuts.

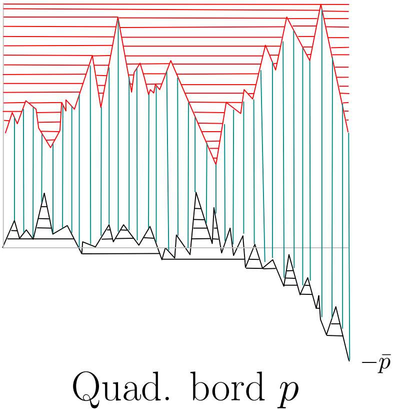
Homeomorphic to $\mathbb{S}^2$.

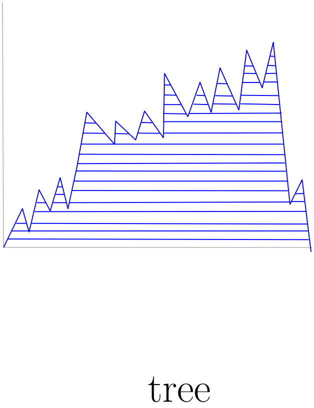
In discrete



Homeomorphic to $\mathbb{S}^2$.

In discrete





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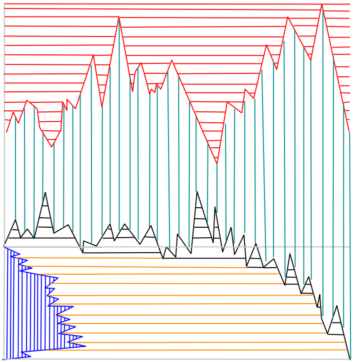

Quad. bord p glued with a tree