Bijections for tree-decorated maps and applications

Luis Fredes
(Work in progress with Avelio Sepúlveda (Univ. Lyon 1))

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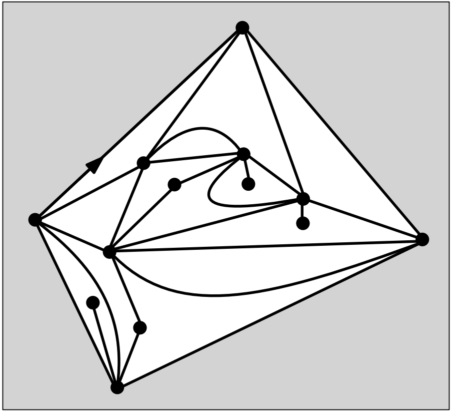
LaBRI, Université de Bordeaux

Overview

MAPS

Map

A **planar map** is a proper embedding of a finite connected planar graph in the sphere, considered up to direct homeomorphisms of the sphere. The **faces** are the connected components of the complement of the edges. It has a distinguished half-edge: the **root edge**. The face that is at the left of the root-edge will be called the **root-face**.



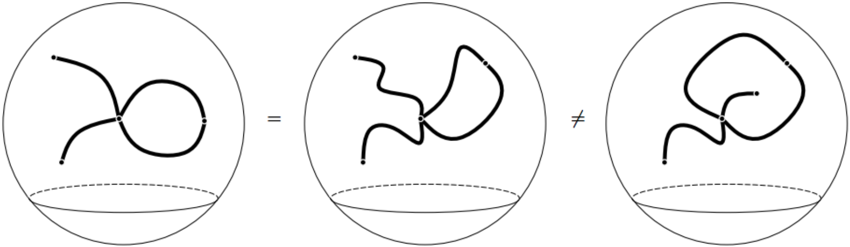


Figure: Same planar graph with different embeddings (sketch by N. Curien).

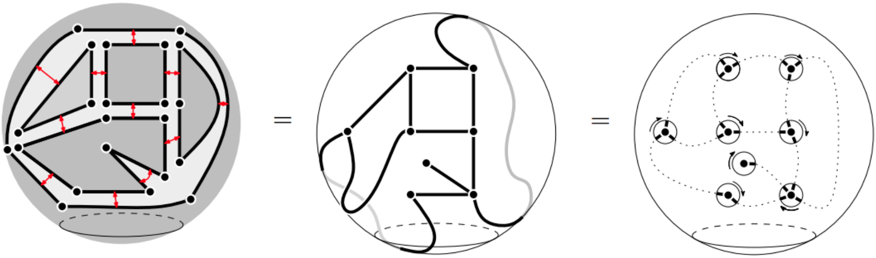
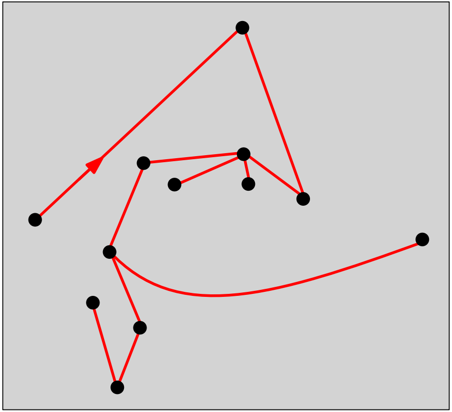


Figure: Same planar map seen as different objects/codings (sketch by N. Curien).

Planar trees

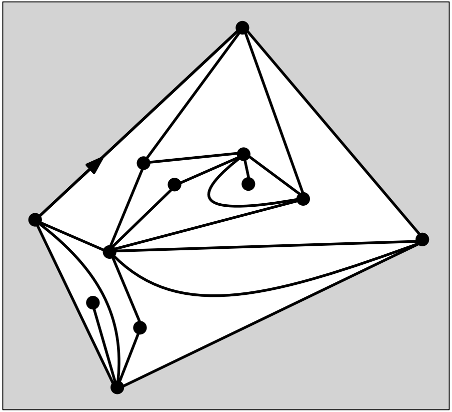
A **planar tree** is a map with one face. Denote as $\mathcal{T}_m$ the number of trees with m edges.

$$\mathcal{T}_m = \mathcal{C}_m = \frac{1}{m+1}\binom{2m}{m}$$



Quadrangulations

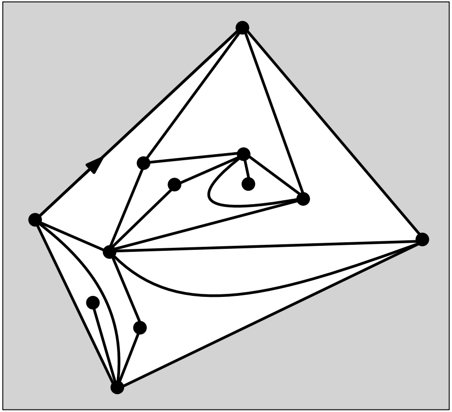
The degree of a face is the number of edges adjacent to it (an edge included in a face is counted twice). A **quadrangulation** is a map whose faces have degree 4.



Quadrangulations

Let $\mathcal{Q}_f$ be the set of all quadrangulations with f faces, then

$$|\mathcal{Q}_f| = 3^f \frac{2}{f+1} \underbrace{\frac{1}{f+1}\binom{2f}{f}}_{\mathcal{C}_f}.$$



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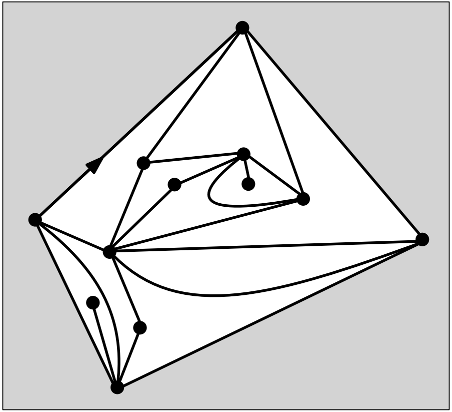
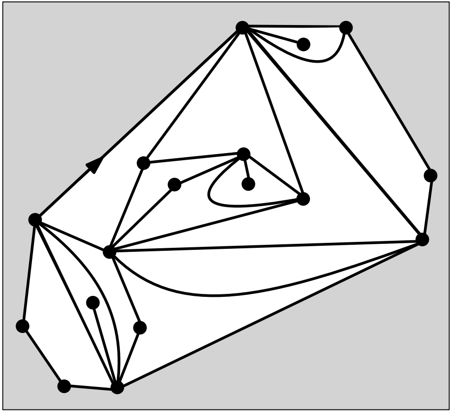
THIS NUMBER ALSO COUNTS GENERAL MAPS WITH $m = f$ EDGES!



Quadrangulations with a boundary

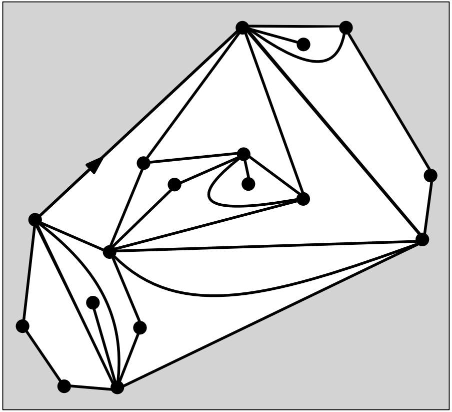
A quadrangulation with a boundary is a map where the **root-face** plays a special role: it has **arbitrary degree**.
All others faces are called **internal faces** and have degree 4.



Quadrangulations with a boundary

The set of quadrangulations with f internal faces and a boundary of size p has cardinality

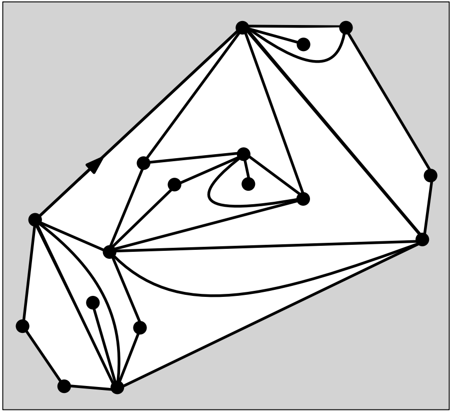
$$3^f \frac{f}{(f+p+1)(2f+p)} \binom{2f+p}{f} \binom{2p}{p}.$$



Quadrangulations with a simple boundary

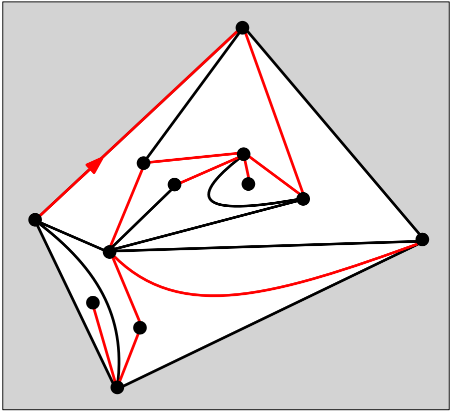
The set of quadrangulations with f internal faces and a simple boundary of size p (root-face of degre p) has cardinality

$$\frac{3^{f-p}(2f+p-1)!}{(f+2p)!(f-p+1)!}\frac{(3p)!}{p!(2p-1)!}.$$



Spanning tree-decorated maps

A spanning tree-decorated map (**ST map**) is a pair (m, t) where m is a map and $t \subset_M \mathrm{m}$ is a spanning tree of m.

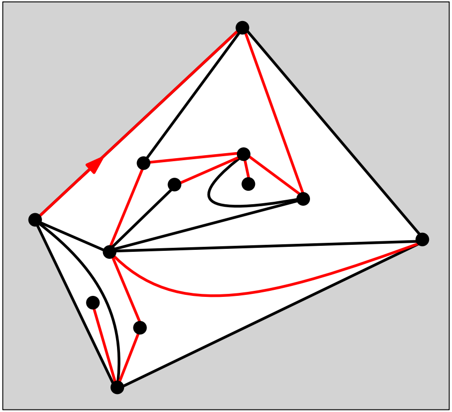


Spanning tree-decorated maps

The family of ST maps with m edges is in bijection with a pair of interlaced trees (mating of trees), one of size m and other of size $m+1$ (lots of bijections for this family). As a consequence this family is counted by

$$\mathcal{C}_m \mathcal{C}_{m+1}$$

TO OUR KNOWLEDGE ST Q-ANGULATIONS HAVE NOT BEEN COUNTED.



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What happens when we use $m = 1$ and $m = f + 1$?

We interpolate between the uniform quadrangulation and the ST quadrangulation!!!!

Tree-decorated map

An (f, m) **tree-decorated map** is a pair (m, t) where m is a map with f faces, and t is a tree with m edges, so that $t \subset_M \mathrm{m}$ containing the root-edge.

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In what follows, a **Uniform** (f, m) **tree-decorated quadrangulations** is a random variable chosen in the family of all (f, m) tree-decorated quadrangulations.

Bijection

Proposition (F. & Sepúlveda '18+)

The set of (f, m) tree-decorated maps is in bijection with the Cartesian product between the set of maps with a simple boundary of size $2m$ and f interior faces and the set of trees with m edges.

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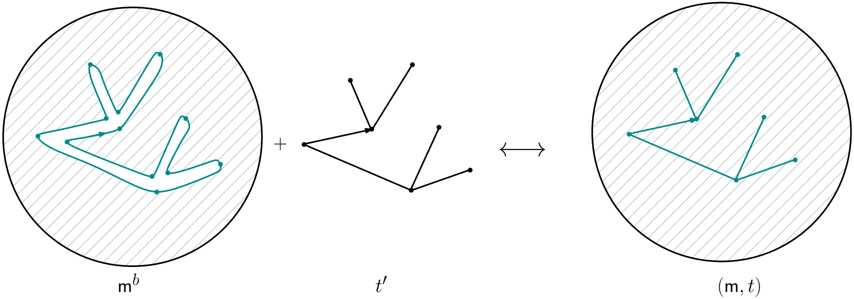

Figure: Sketch of the bijection. Left: Map with boundary and planted tree representing this bijection. Right: Tree decorated map. We plot it being embedded in the sphere. The arrows are root-edges and the grid lines represent the inner faces.

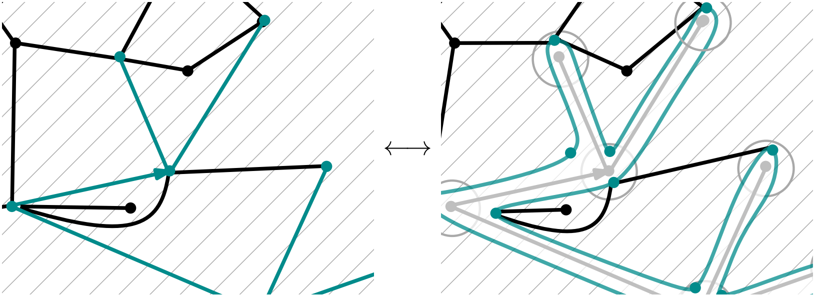

Figure: Left: Zoom of the tree decorated map. In green the decoration and in black the edges that do not belong to the decoration.

Right: Map with boundary and planted tree. Transformation obtained from the corners (green points) of the decoration.

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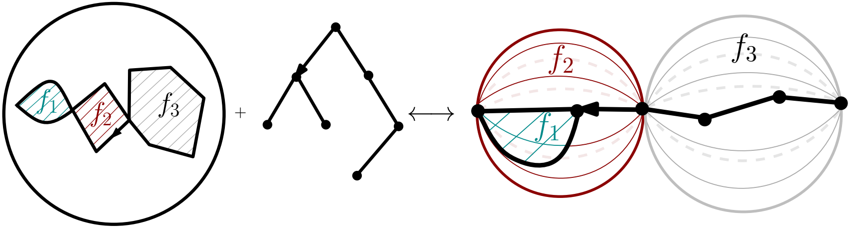

Figure: Left: Map with a non-simple boundary (interior faces filled with lines) and a tree. Right: Bubbles (3D plot) form by the gluing of a map with non-simple boundary and a tree.

Counting results

Corollary (F. & Sepúlveda '18+)

The number of (f, m) *tree-decorated triangulations are*

$$\frac{2^{f-2m}(3f/2+m-2)!!}{(f/2-m+1)!(f/2+3m)!!}2m\binom{4m}{2m}\frac{1}{m+1}\binom{2m}{m}, \tag{1}$$

where $n!! = \prod_{i=0}^{\lfloor (n-1)/2 \rfloor}(n-2i)$.
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$$3^{f-m}\frac{(2f+m-1)!}{(f+2m)!(f-m+1)!}\frac{(3m)!}{m!(2m-1)!}\frac{1}{m+1}\binom{2m}{m} \tag{2}$$

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We also count

- Maps (triangulations and quadrangulations) with a simple boundary decorated in a spanning tree.
- Maps (triangulations and quadrangulations) with a simple boundary decorated in a subtree.
- Forest-decorated maps.

CONVERGENCE RESULTS

Local Limit (Benjamini-Schramm Topology '01)

For a map m and $r \in \mathbb{N}$, define $B_r(\mathrm{m})$ as the ball of radius r from the root-vertex. Consider $\mathcal{M}$ a family of finite maps. The local topology on $\mathcal{M}$ is the metric space $(\mathcal{M}, \mathrm{d_{loc}})$, where

$$\mathrm{d_{loc}}(\mathrm{m_1}, \mathrm{m_2}) = (1 + \sup\{\mathrm{r} \geq 0 : \mathrm{B_r}(\mathrm{m_1}) = \mathrm{B_r}(\mathrm{m_2})\})^{-1}$$

Meaning that a sequence of maps $(\mathrm{m}_i)_{i \in \mathbb{N}}$ converges if for all $r \in \mathbb{N}$, $B_r(\mathrm{m}_i)$ is constant from certain point on.

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Proposition

The space $(\overline{\mathcal{M}}, \mathrm{d_{loc}})$ is Polish (metric, separable and complete).

Gromov-Hausdorff topology

Recall that if (E, d_E) is a metric space and $A, B \subset Z$, the Hausdorff distance between A and B is given by

$$\mathrm{d_H}(\mathrm{A}, \mathrm{B}) = \max\left\{ \max_{\mathrm{x}\in \mathrm{B}} \mathrm{d_E}(\mathrm{x}, \mathrm{A}), \max_{\mathrm{y}\in \mathrm{A}} \mathrm{d_E}(\mathrm{y}, \mathrm{B}) \right\}$$

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Consider the set S of compact metric spaces up to isometry classes. The Gromov-Hausdorff distance between two metric spaces (X, d) and (X', d') is defined as

$$\mathrm{d_{GH}}((\mathrm{X}, \mathrm{d}), (\mathrm{X}', \mathrm{d}')) = \inf \mathrm{d_H}(\phi(\mathrm{X}), \phi'(\mathrm{X}'))$$

where the infimum is taken over all metric spaces $(E, \mathrm{d_E})$ and all isometric embeddings ϕ, ϕ' from X, X' respectively into E.

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Proposition

The function $\mathrm{d_{GH}}$ *induces a metric on* S*. The space* $(S, \mathrm{d_{GH}})$ *is separable and complete.*

Uniform Trees

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- *Each vertex has bounded degree.*
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$$\left(t_m, \frac{\mathrm{d}_{\mathrm{Tree}}}{m^{1/2}}\right) \xrightarrow[GH]{(d)} CRT$$

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Proposition

- *The CRT is a tree.*
- *Almost every point is a leaf.*
- *Hausdorff dimension 2.*
- *Its geodesics are represented in the coding.*

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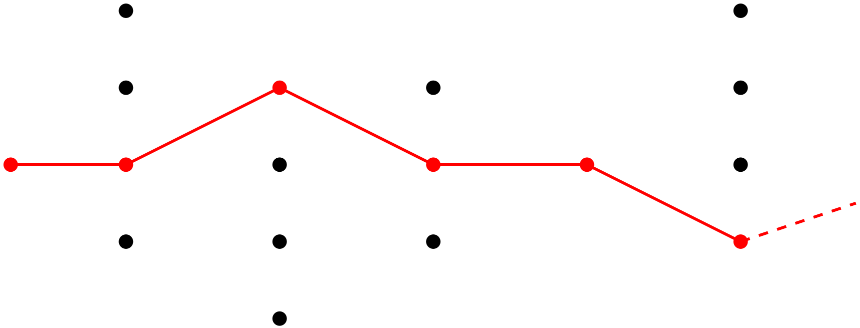

Figure: Geometry of t_∞.

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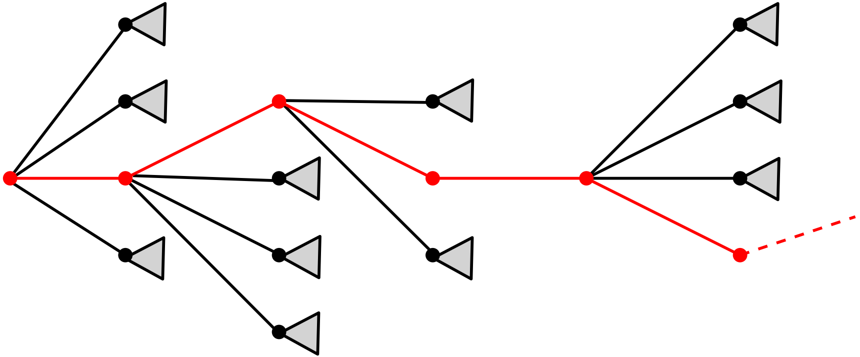

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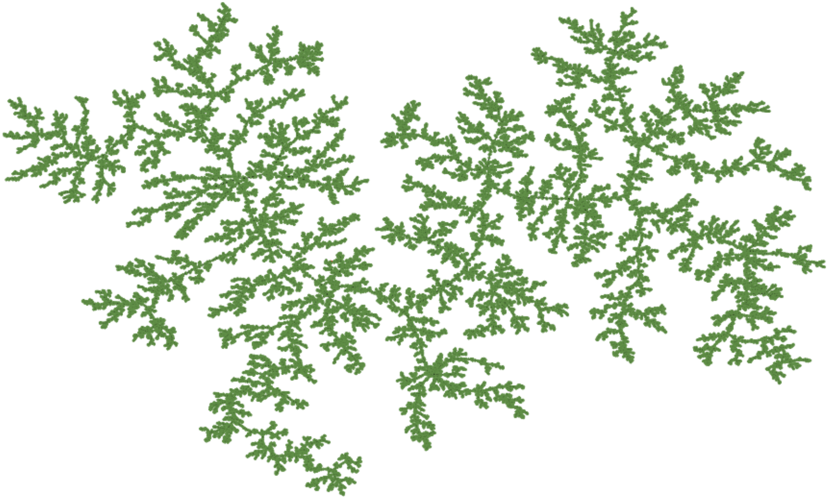

Figure: Uniform random tree 50k edges.

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Properties

- *The UIPQ is an infinite quadrangulation.*
- *Properties about the volume and perimeter of the exploration of the UIPQ are known.*
- *Locally finitely many faces.*

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Properties

- *Hausdorff dimension is 4 (Le Gall '07).*
- *Homeomorphic to the two dimensional sphere (Le Gall & Paulin '08).*
- *Its geodesics are described by the coding (well labeled trees).*

UIPT

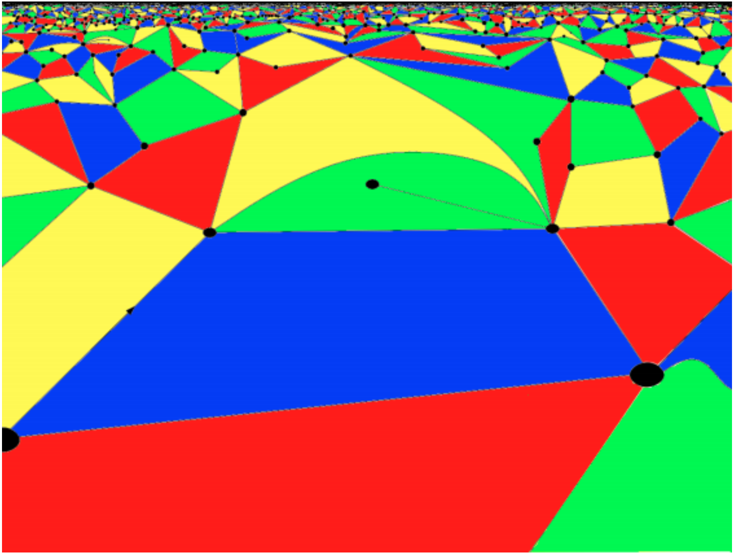

Figure: UIPT representation.
(Sketch by N. Curien)

Brownian map

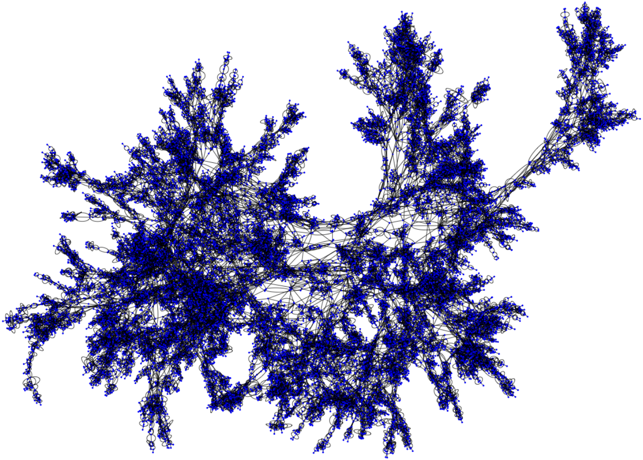

Figure: Brownian map 30k faces.

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Theorem (Curien & Miermont '12)

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Properties

- q_∞^p *is called the Uniform Infinite Planar Quadrangulation with a boundary of perimeter* p.
- *They also obtain the convergences above conditioned to have simple boundary*
- *The* q_∞^p *has one infinite irreducible component, called the core. Moreover,*

$$\frac{\partial Core(\mathrm{q}_\infty^p)}{2p} \xrightarrow[p\to\infty]{(prob)} \frac{1}{3}$$

.

UIHPQ

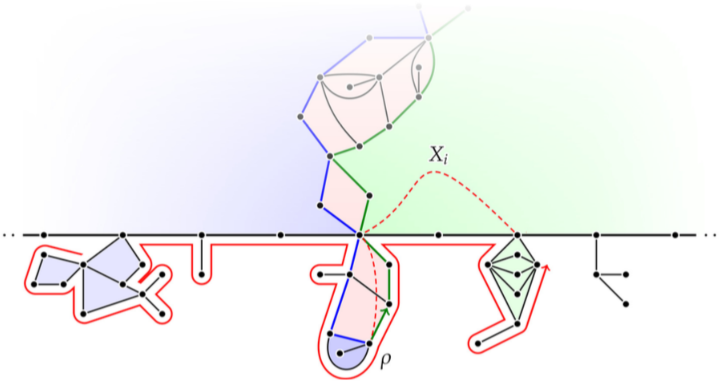

Figure: UIHPQ
(Sketch by N. Curien and A. Caraceni).

Brownian Disk

Let q_f^p be a map uniformly chosen in the set of quadrangulations with f faces and boundary of size p. For a sequence $(p_n)_{n\in\mathbb{N}}$, define $\bar{p} = \lim p_n n^{-1/2}$ as $n \to \infty$.

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Theorem (Scaling limit (Betinelli '15))

$$\left(\mathrm{q}_n, \frac{\mathrm{d}_{\mathrm{map}}}{s(f,p_f)}\right) \xrightarrow[GH]{(d)} \begin{cases} \textit{Brownian map} & \textit{if } s(f,p_f) = f^{1/4} \textit{ and } \bar{p} = 0 \\ \textit{Brownian disk} & \textit{if } s(f,p_f) = f^{1/4} \textit{ and } \bar{p} \in (0,+\infty) \\ \textit{CRT} & \textit{if } s(f,p_f) = 2p_f^{1/2} \textit{ and } \bar{p} = \infty \end{cases}$$

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Properties (Betinelli & Miermont '15)

Brownian disk properties

- *The boundary is simple.*
- *Hausdorff dimension 4 in the interior, 2 in the boundary.*
- *Homeomorphic to the two dimensional disk.*
- *Links with the Brownian map.*

Brownian Disk

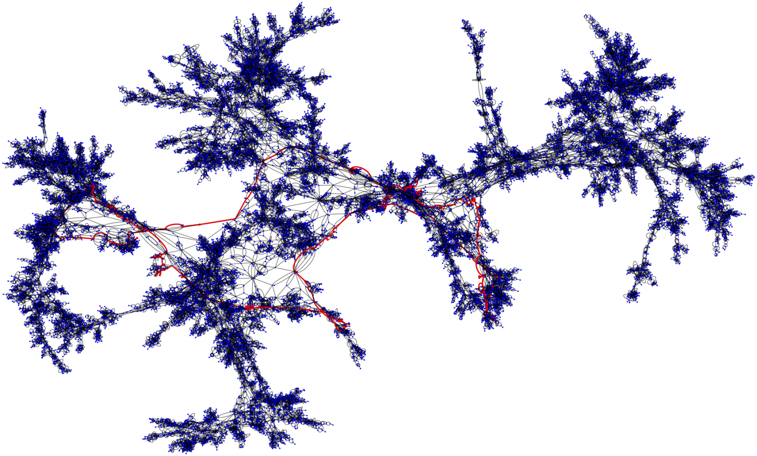

Figure: Uniform quadrangulation with a boundary 30k interior faces, 173 edges in the boundary.

Uniform ST map

Let $q_f^{\mathcal{ST}}$ be uniformly chosen in the set of ST quadrangulations with f faces.

- The conjectured scaling limit of these objects should be related to continuum Liouville quantum gravity.
- Recently it has been shown that there exists a constant $0.275 \leq \chi \leq 0.288$, such that the expected diameter is of order n^{χ} (Ding & Gwynne '18, Gwynne, Holden & Sun '16).
- In the case of convergence as a metric space, there is evidence that the limit is not the Brownian map.
- There exists a local limit for this object and other decorated-families (Sheffield '11).

Uniform ST decorated-quadrangulation "Scaling limit"

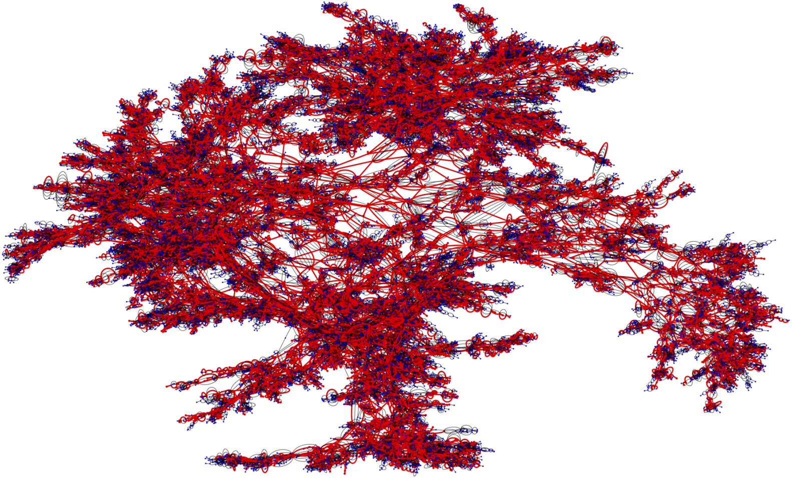

Figure: Uniform spanning tree-decorated quadrangulation 30k faces.

Motivation

1. Uniform (f, m) tree-decorated quadrangulation model is that it interpolates between uniform quadrangulation with f faces and the uniform ST-decorated quadrangulation with f faces.
   In light of this effect, we hope to give a phase transition between these objects obtaining an insight about the metric scaling limit of uniform ST map.

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1. Uniform (f, m) tree-decorated quadrangulation model is that it interpolates between uniform quadrangulation with f faces and the uniform ST-decorated quadrangulation with f faces.
   In light of this effect, we hope to give a phase transition between these objects obtaining an insight about the metric scaling limit of uniform ST map.
2. This model could encode two different statistical mechanic objects, one on the tree and one on the map without considering the tree.

Local limit results

What happens if we glue the $\mathrm{UIHPQ_S}$ (local limit of quadrangulations with a simple boundary) with t_∞ (local limit of uniform trees)?

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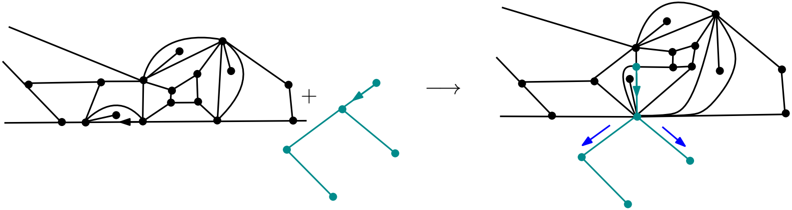

Figure: First step in the sequential gluing procedure. The second step is sketched with the next edges in the contour to glue in blue.

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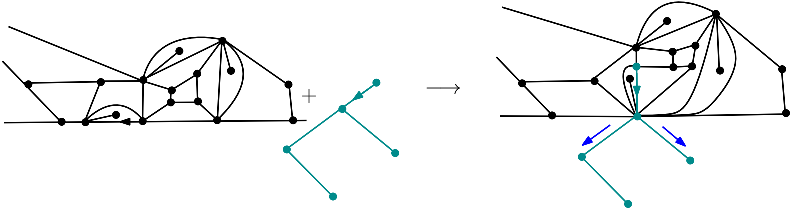

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Proposition (F. & Sepúlveda '18+)

There exists a local limit for the gluing of an infinite tree t with a **UIHPQ**$_S$.

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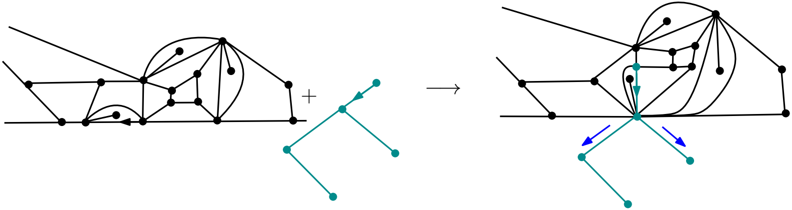

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Remark

We obtain more local limits.

Scaling limit results

Corollary (F. & Sepúlveda '18+)

Let (m_f, t_{m_f}) *be a uniform* (f, m_f) *tree-decorated quadrangulations with* $m_f \leq f+1$. *Then as* $m_f \to \infty$,

$$\left(t_{m_f}, \frac{\mathrm{d}_{\mathrm{t}_{\mathrm{m_f}}}}{m_f^{1/2}} \right) \xrightarrow[GH]{(d)} CRT.$$

Scaling limit conjecture

Conjecture (F. & Sepúlveda '18+)

Let $(\mathrm{m}_f, t_{\mathrm{m}_f})$ *be a uniform* (f, m_f) *tree-decorated quadrangulation with* $m_f = O(f^{\alpha})$*. Depending on* α

$$\left((\mathrm{m}_f, t_{\mathrm{m}_f}), \frac{\mathrm{d}_{\mathrm{m_f}}}{f^{\beta}}\right) \xrightarrow[GH]{(d)} \begin{cases} \textit{Brownian map} & \textit{if } \alpha \leq 1/2, \beta = 1/4 \textit{(Proved)} \\ \textit{Shocked map} & \textit{if } \alpha = 1/2, \beta = 1/4 \textit{(In progress)} \\ \textit{Tree-decorated map} & \textit{if } \alpha \geq 1/2, \\ & \beta = \left(2\chi - \frac{1}{2}\right)\alpha - \chi + \frac{1}{2} \end{cases}$$

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- The Shocked map is not trivial (Proved).
- The Shocked map should be the gluing between a Brownian disk with perimeter p and a CRT.
- The Shocked map has Hausdorff dimension 4 outside the tree (Proved).
- It should have dimension 2 on the decoration (In progress).
- Homeomorphic to the two dimensional sphere (In progress).

Shocked map

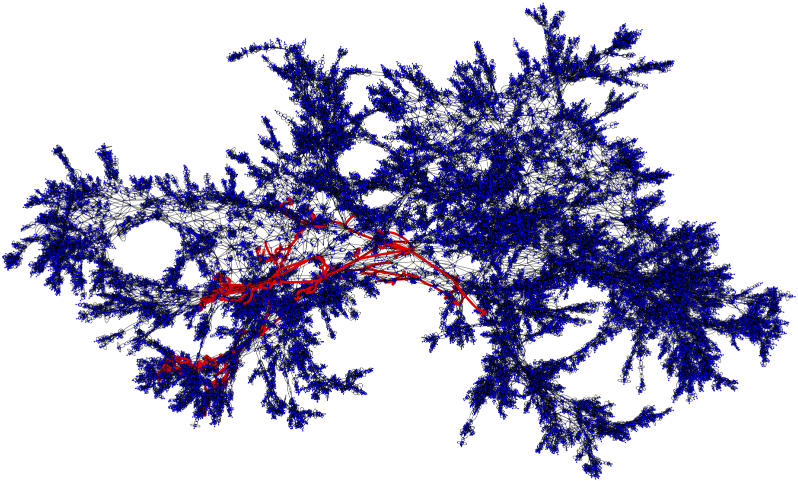

Figure: Uniform tree-decorated quadrangulation 90k faces decorated on a tree of size 500.

Why shocked?

Figure: Golf field struck by lightning.

Thank you!