Bijections for tree-decorated map and applications to random maps.

Luis Fredes
(Work in progress with Avelio Sepúlveda (Univ. Lyon 1))

IRIF, 2019

Tree-decorated planar maps: combinatorial results.

with A. Sepúlveda.

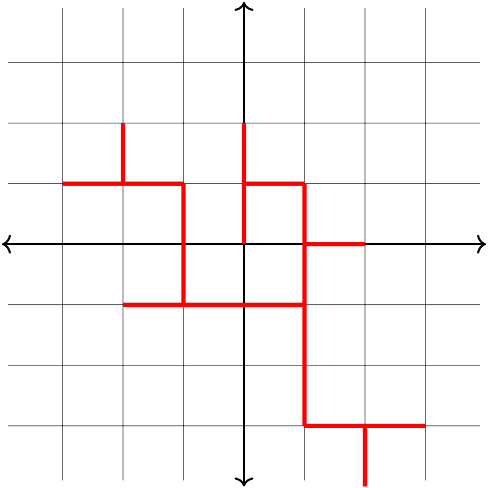

Figure: Uniform random tree of size 20 containing the origin on $\mathbb{Z}^2$.

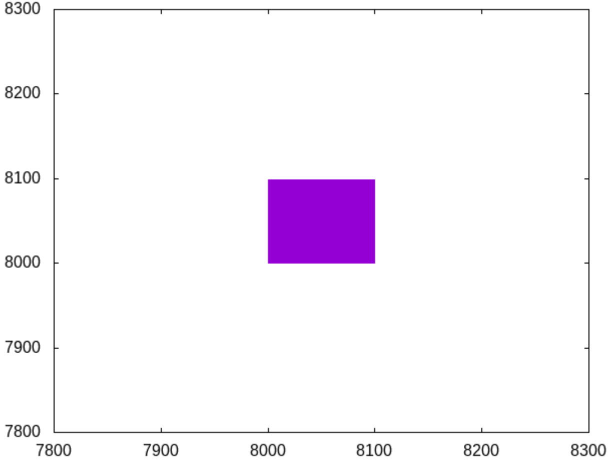

Figure: Dynamic on trees of size 10000.

Tree-decorated planar maps: combinatorial results.

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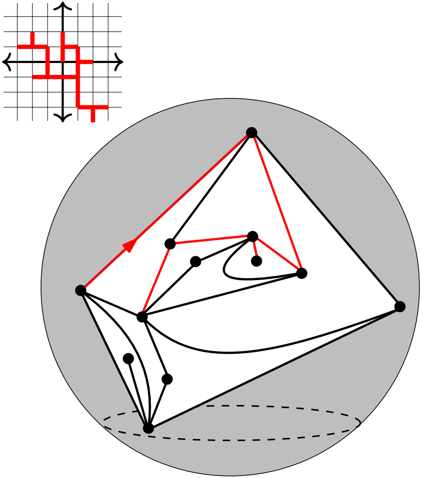
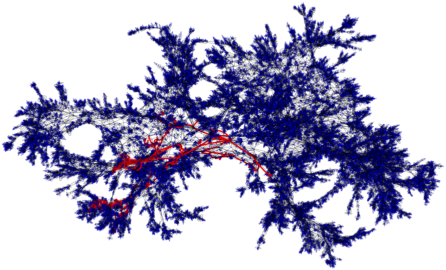


(a) tree-decorated quad. 10 faces, tree of size 6.



(b) Unif. tree-decorated quad. 90k faces and tree of size 500.

Maps

- A **face**= A connected component of the complement of the edges.
- The **root-edge**= distinguished half edge.
- The **root-face**= face to the left of the root-edge.
- **Degree of a face**= number of adjacent edges to it.

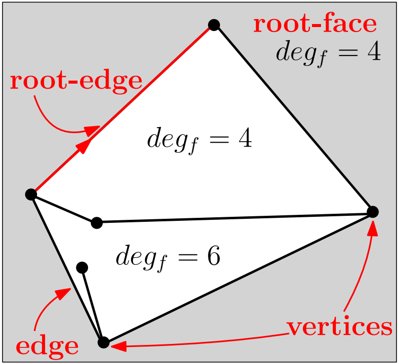
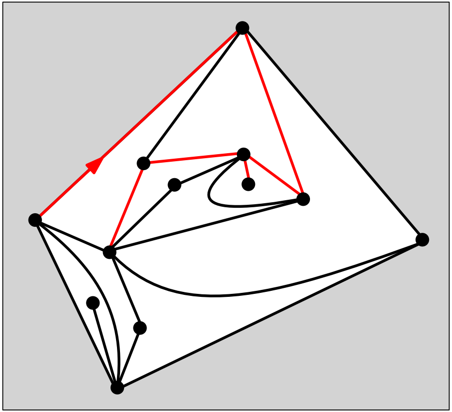


Figure: Same graph, different embeddings on the sphere.

A (f, a) **tree-decorated map** is a pair $(\mathfrak{m}, \mathfrak{t})$ where:

- $\mathfrak{m}$ is a rooted map with f faces.
- $\mathfrak{t}$ is a submap of $\mathfrak{m}$ ($\mathfrak{t} \subset_M \mathfrak{m}$).
- $\mathfrak{t}$ is a tree with a edges.
- $\mathfrak{t}$ contains **the root-edge of $\mathfrak{m}$**.



~~Spanning~~ tree-decorated maps

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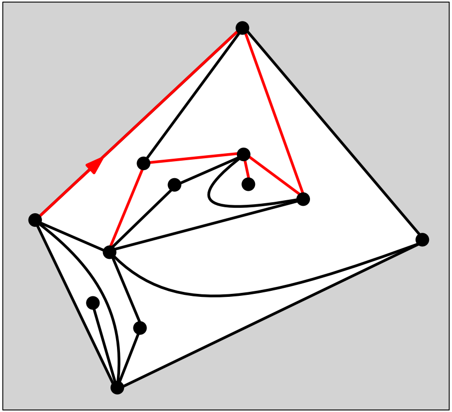


It interpolates: In the case of quadrangulations

- $\underline{a = 1 \rightarrow}$ quadrangulations with f faces. **[Tutte '60, Bender & Canfield '94, Cori-Vauquelin-Schaeffer '98, Schaeffer '97, Bettinelli '15]**
- $\underline{a = f + 1 \rightarrow}$ spanning-tree decorated quadrangulations with f faces. **[Mullin '67, Walsh & Lehman '72, Cori et al '86; Bernardi '06]**

Counting results

Theorem (F. & Sepúlveda '19)

The number of (f, a) tree-decorated quadrangulations is

$$3^{f-a}\frac{(2f+a-1)!}{(f+2a)!(f-a+1)!}\frac{2a}{a+1}\binom{3a}{a,a,a}$$

We also count

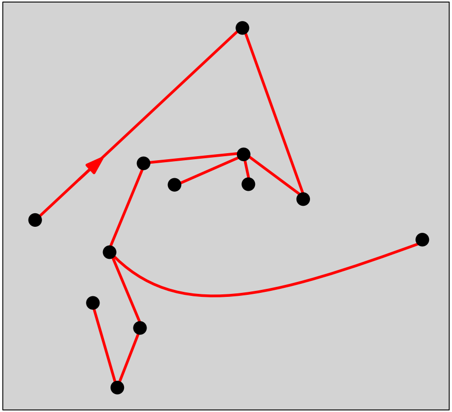
- (f, a) tree-decorated triangulations.
- Maps (triangulations and quadrangulations) with a simple boundary decorated in a subtree.
- Forest-decorated maps.
- "Tree-decorated general maps".

Planar trees

A **planar tree** is a rooted map with one face.

Number of planar trees with a edges

$$\mathcal{C}_a = \frac{1}{a+1}\binom{2a}{a}.$$

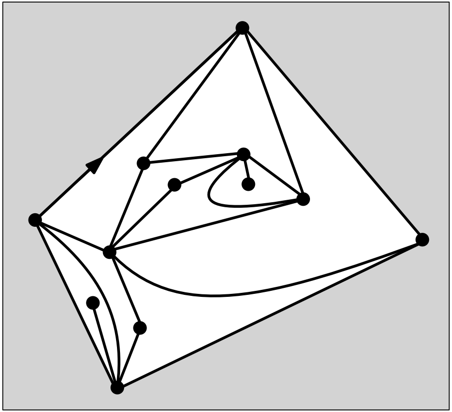


Quadrangulation: map whose faces have degree 4.

Number of quadrangulations with f faces

$$3^f \frac{2}{f+2} \underbrace{\frac{1}{f+1}\binom{2f}{f}}_{\mathcal{C}_f}.$$

Analytic [Tutte '60] and Bijective [Cori-Vauquelin-Schaeffer '98].



Quadrangulations with a boundary

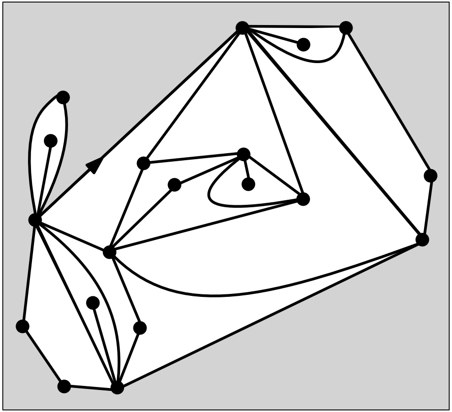
Quadrangulation with a boundary: All faces, but the root-face, have degree 4.

Number of quadrangulations with a boundary with:

- f internal faces.
- **boundary** of size $2p$.

$$\frac{3^f p}{(f+p+1)(f+p)} \binom{2f+p-1}{f} \binom{2p}{p}.$$

Analytic by [Bender & Canfield '94; Bouttier & Guitter '09] and bijective by [Schaeffer '97 ; Bettinelli '15]

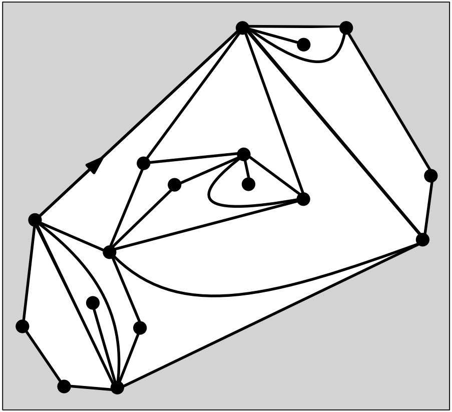


Number of quadrangulations with a simple boundary with:

- f internal faces.
- **simple boundary** of size $2p$ (root-face of degree $2p$).

$$\frac{3^{f-p}2p}{(f+2p)(f+2p-1)}\binom{2f+p-1}{f-p+1}\binom{3p}{p}.$$

Analytic [Bouttier & Guitter '09] and bijective [Bernardi & Fusy '17].



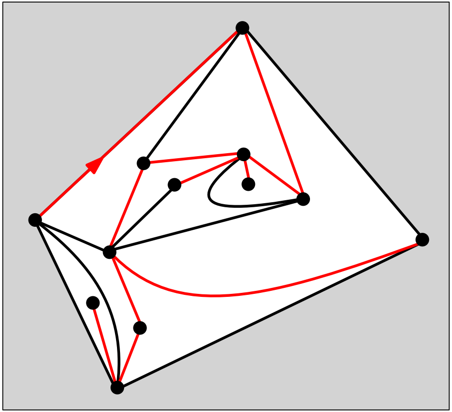
Spanning tree-decorated map (ST map): is a pair $(\mathfrak{m}, \mathfrak{t})$ where:

- $\mathfrak{m}$ is a rooted-map.
- $\mathfrak{t}$ is a submap of $\mathfrak{m}$ ($\mathfrak{t} \subset_M \mathfrak{m}$).
- $\mathfrak{t}$ is a spanning tree of $\mathfrak{m}$.

Number of ST maps with a edges

$$\mathcal{C}_a \mathcal{C}_{a+1}$$

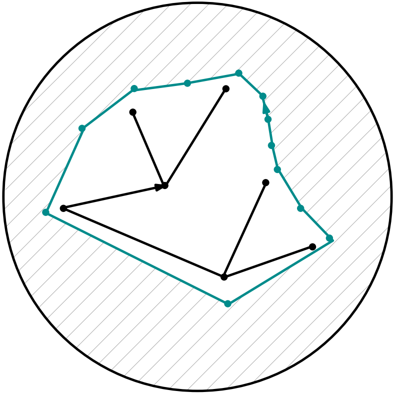
Analytic by [Mullin '67] and bijective by [Walsh and Lehman '72; Cori, Dulucq & Viennot '86; Bernardi '06]

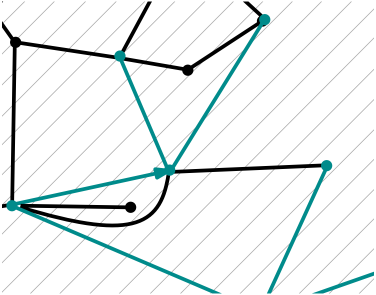


Bijection

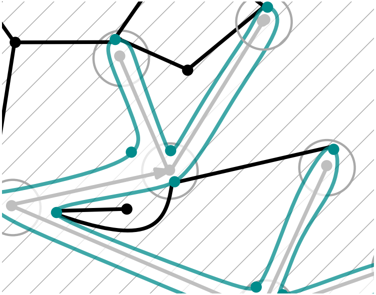
Theorem (F. & Sepúlveda '19)

The set of (f, a) tree-decorated maps is in bijection with
(the set of maps with a simple boundary of size $2a$ and f interior faces)
$\times$ (the set of trees with a edges).



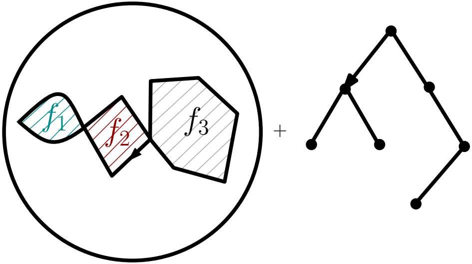


$\longleftrightarrow$



What do we obtain when the boundary is not simple?

For bridgeless maps it gives BUBBLE-MAPS!



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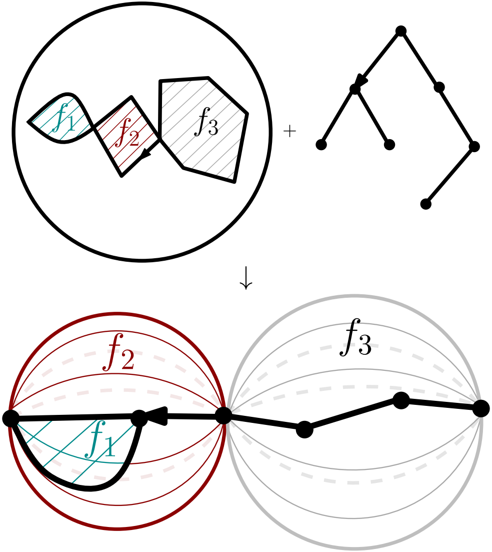
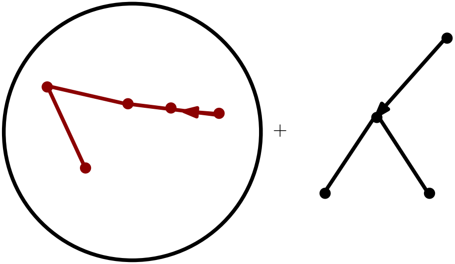


Degenerate map!



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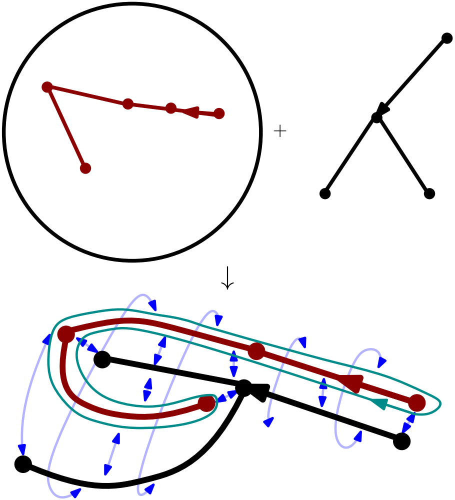


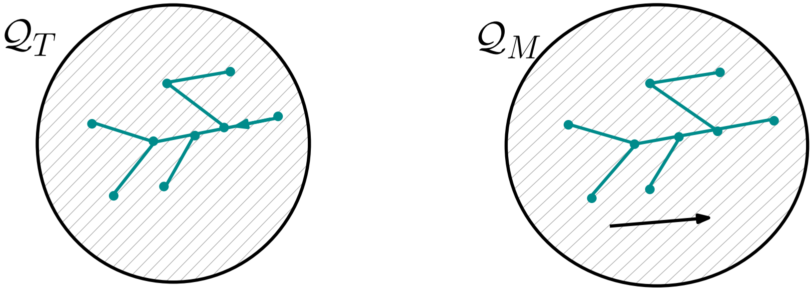
Remarks and extensions

The bijection makes a correspondence between:

[Tree-decorated map]		**[Map with a boundary, Tree]**
Faces of degree q	$\longleftrightarrow$	Internal faces of degree q
Internal vertices of degree d	$\longleftrightarrow$	Internal vertices of degree d
Internal edges	$\longleftrightarrow$	Internal edges
Corner of the tree	$\longleftrightarrow$	Boundary vertices.

- We can restrict the bijection to q-angulations.
- It can be restricted to some subfamilies of trees:
  1. Binary tree-decorated Maps.
  2. SAW decorated maps (Already done by Caraceni & Curien).

Re-rooting



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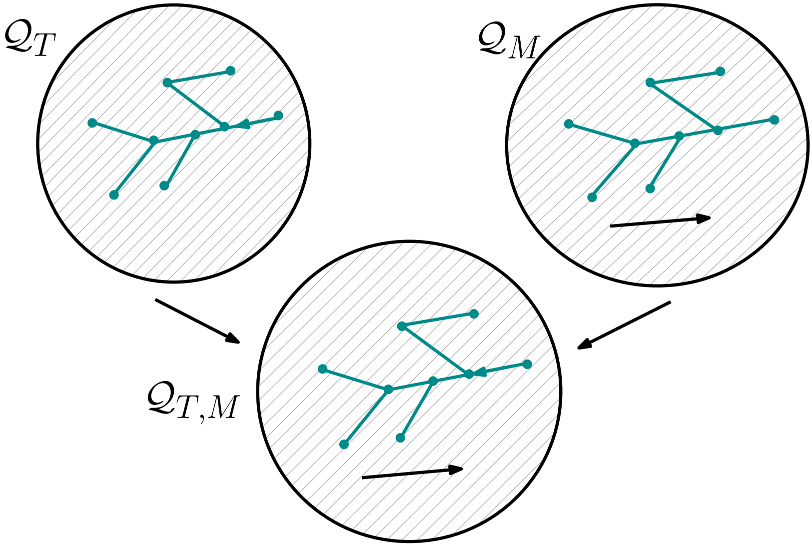

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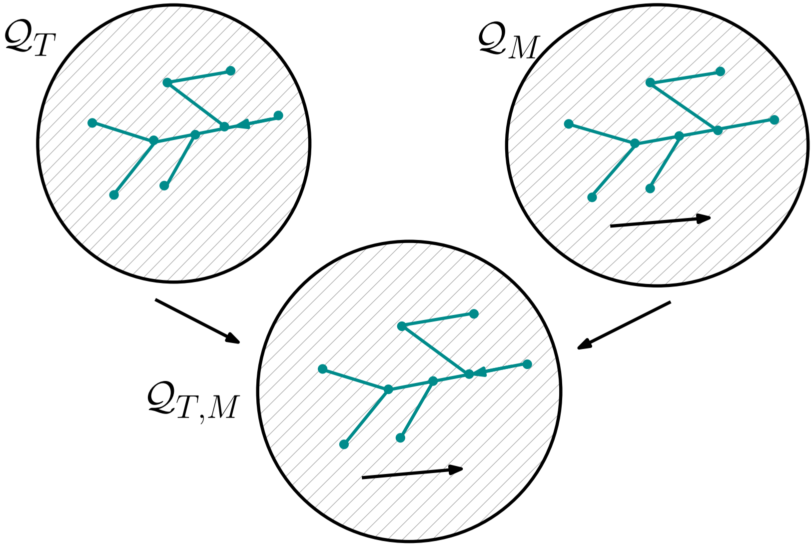

$$|\mathcal{Q}_T| \times 2|E| = |\mathcal{Q}_M| \times 2|T|$$

Counting results

In the case of spanning tree decorated quadrangulations rooted in the tree we obtain

$$\mathcal{C}_{2,f} = \frac{2}{(f+1)(f+2)}\binom{3f}{f,f,f}$$

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A possible generalization of Catalan numbers:

$$\mathcal{C}_{m,n} = m!\left(\prod_{i=1}^{m}\frac{1}{(n+i)}\right)\binom{(m+1)n}{\underbrace{n,n,\ldots,n}_{m+1 \text{ times}}} = \binom{m+n}{n}^{-1}\binom{(m+1)n}{\underbrace{n,n,\ldots,n}_{m+1 \text{ times}}}$$

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Proposition

$\mathcal{C}_{m,n}$ is an integer $\forall n, m$.

Proof by D. Sénizergues.

Define $A_{n,m} = \#$ standard young tableaux of shape $\lambda = \underbrace{(n, n, ..., n)}_{m \text{ times}}$.

From the hook-length formula we see that

$$C_{m,n} = \left(\prod_{i=1}^{m-1} \binom{n+i}{i} \right) \times A_{n,m+1}$$

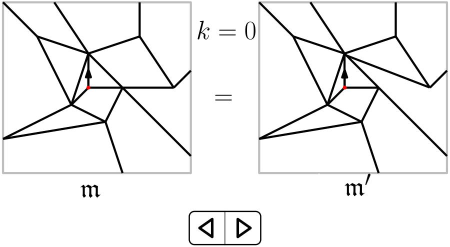
$\square$

Consider:

- $B_r(\mathfrak{m})$ = ball of radius r from the root-vertex.
- $\mathcal{M}$= set of (locally finite) maps.

We endowed $\mathcal{M}$ with the (local) topology induced by

$$\mathrm{d_{loc}}(\mathfrak{m}_1, \mathfrak{m}_2) = (1 + \sup\{\mathrm{r} \geq 0 : \mathrm{B_r}(\mathfrak{m}_1) = \mathrm{B_r}(\mathfrak{m}_2)\})^{-1}$$



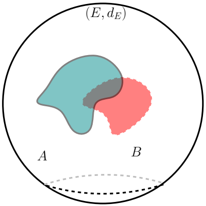
Proposition

The space $(\overline{\mathcal{M}}, \mathrm{d_{loc}})$ *is Polish (metric, separable and complete).*

Gromov-Hausdorff topology

Let (E, d_E) be a metric space and $A, B \subset E$. The **Hausdorff distance** is

$$d_H(A, B) = \inf \left\{ \varepsilon > 0 : A \subset B_\varepsilon, B \subset A_\varepsilon \right\}$$



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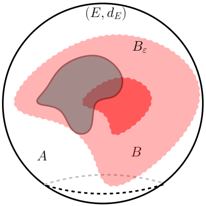
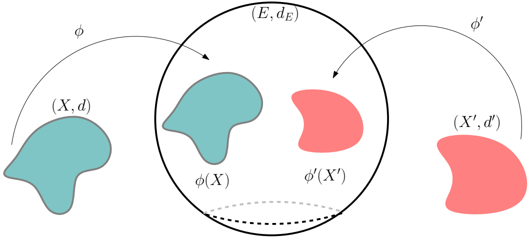


Gromov-Hausdorff topology

Consider the set S of compact metric spaces up to isometry classes. The **Gromov-Hausdorff distance** between two metric spaces (X, d) and (X', d') is defined as

$$\mathrm{d_{GH}}((\mathrm{X}, \mathrm{d}), (\mathrm{X}', \mathrm{d}')) = \inf \mathrm{d_H}(\phi(\mathrm{X}), \phi'(\mathrm{X}'))$$

where the infimum is taken over all metric spaces $(E, \mathrm{d_E})$ and all isometric embeddings ϕ, ϕ' from X, X' respectively into E.



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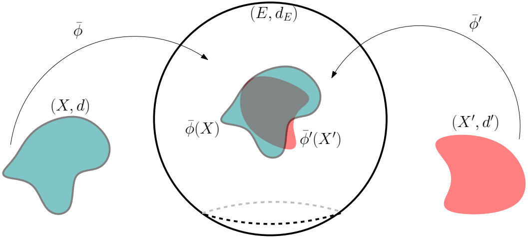

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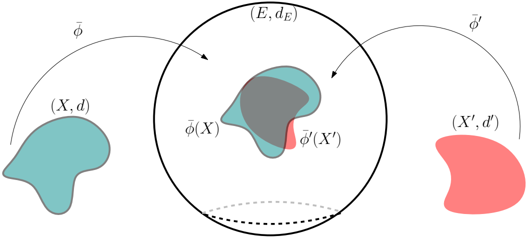

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Proposition

The function $\mathrm{d_{GH}}$ induces a metric on S. The space $(S, \mathrm{d_{GH}})$ is separable and complete.

Uniform Trees

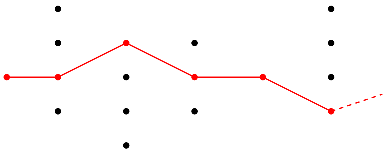
$\mathfrak{t}_{\mathbf{a}}$= Unif. tree with a edges.

Theorem (Kesten '86)

$$\mathfrak{t}_a \xrightarrow[local]{(d)} \mathfrak{t}_\infty$$

Properties

- $\mathfrak{t}_\infty$ *is an infinite tree.*
- *It has one infinite branch (the spine).*



$\mathfrak{t}_\infty$ **construction.**

Uniform Trees

$\mathfrak{t}_{\color{blue}{a}}$= Unif. tree with $\color{blue}{a}$ edges.

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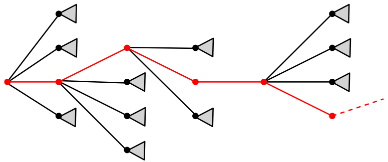

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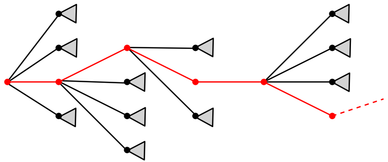
Theorem (Aldous '91)

$$\left(\mathfrak{t}_a, \frac{d_{Tree}}{a^{1/2}}\right) \xrightarrow[GH]{(d)} CRT$$

Properties

- *The CRT is a tree.*
- *Almost every point is a leaf.*
- *Hausdorff dimension 2.(Duquesne & Le Gall '05)*



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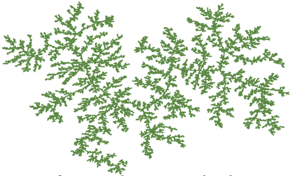


Uniform random tree 50k edges.

Uniform quadrangulations

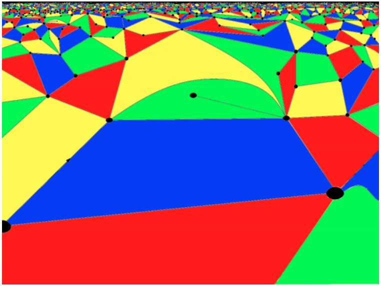
$\mathfrak{q}_{\mathsf{f}}$= Unif. quadrangulation with f faces.

Theorem (Krikun '06)

$$\mathfrak{q}_f \xrightarrow[local]{(d)} UIPQ$$

Properties

- *The UIPQ is an infinite quad.*



(Sketch by N. Curien)

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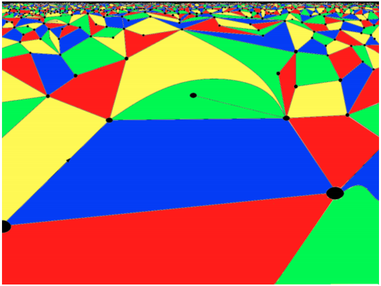


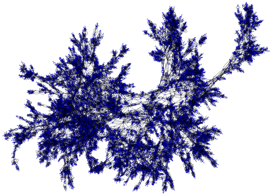
(Sketch by N. Curien)

Theorem (Miermont '13, Le Gall '13)

$$\left(\mathfrak{q}_f, \frac{\mathrm{d}_{\mathrm{map}}}{f^{1/4}}\right) \xrightarrow[GH]{(d)} \textit{Brownian map}$$

Properties

- *Hausdorff dim. is 4 (Le Gall '07).*
- *Homeomorphic to $\mathbb{S}^2$ (Le Gall & Paulin '08).*



Unif. quadrangulation 30k faces.

Uniform quadrangulation with a boundary: local limit

$\mathfrak{q}_{\mathbf{f,p}}$= Unif. quadrangulations with a boundary of size $2p$ and f faces.

Theorem (Curien & Miermont '12)

$$\mathfrak{q}_{f,p} \xrightarrow[local(f\to\infty)]{(d)} \mathfrak{q}_{\infty,p} \xrightarrow[local(p\to\infty)]{(d)} UIHPQ$$

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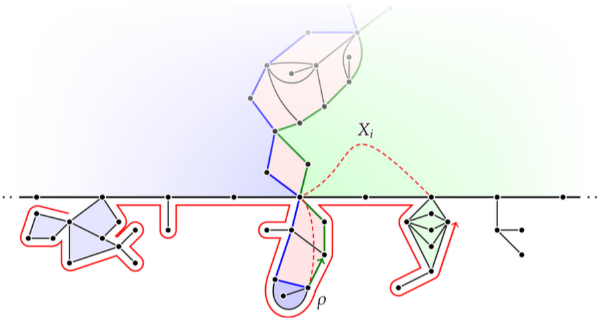


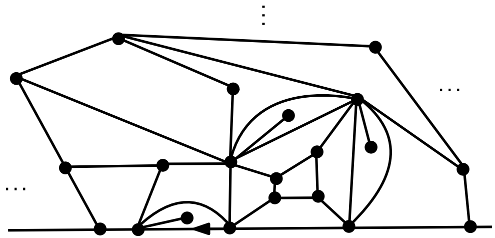
UIHPQ (sketch by N. Curien & A. Caraceni) .

Uniform quadrangulation with a simple boundary: local limit

$\mathfrak{q}^{\mathbf{S}}_{\mathbf{f,p}}$= Unif. quadrangulations with a **simple** boundary of size $2p$ and f faces.

Theorem (Curien & Miermont '12)

$$\mathfrak{q}^{S}_{f,p} \xrightarrow[local(f\to\infty)]{(d)} \mathfrak{q}^{S}_{\infty,p} \xrightarrow[local(p\to\infty)]{(d)} UIHPQ^{S}$$



sketch of a $UIHPQ^{S}$.

Uniform quadrangulation with a boundary: GH limit

$\mathfrak{q}_{\mathbf{f},\mathbf{p}}$= Unif. quadrangulations with a boundary of size $2p$ and f faces. For a sequence $(p(f))_{f\in\mathbb{N}}$, define $\overline{p} = \lim p(f)f^{-1/2}$ as $f \to \infty$.

Theorem (Scaling limit (Bettinelli '15))

$$\left(\mathfrak{q}_{f,p(f)}, \frac{\mathrm{d}_{\mathrm{map}}}{s(f,p(f))}\right) \xrightarrow[GH]{(d)} \begin{cases} \textit{Brownian map} & \textit{if } s(f,p(f)) = f^{1/4} \textit{ and } \overline{p} = 0 \\ \textbf{\textit{Brownian disk}} & \textit{if } s(f,p(f)) = f^{1/4} \textit{ and } \overline{p} \in (0,+\infty) \\ \textit{CRT} & \textit{if } s(f,p(f)) = 2p(f)^{1/2} \textit{ and } \overline{p} = \infty \end{cases}$$

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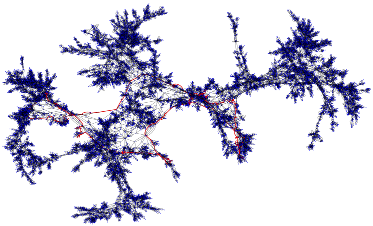
Properties (Bettinelli & Miermont '15)

Brownian disk properties

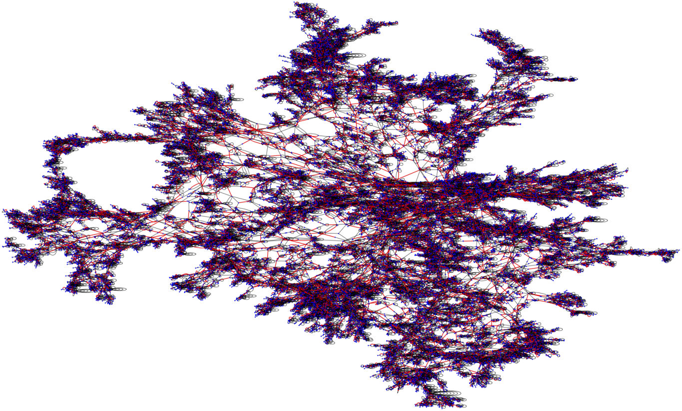
- *The boundary is simple.*
- *Hausdorff dim. 4 in the interior, 2 in the boundary.*
- *Homeomorphic to the disk 2d.*



Unif. quad. with 30k interior faces and boundary 173.

Uniform ST map

- Convergence for the local topology (Sheffield '11).
- The limit (if it exists) seems not to the Brownian map.
- Expected diameter is of order n^{χ} for $0.275 \leq \chi \leq 0.288$ (Ding & Gwynne '18, Gwynne, Holden & Sun '16).



Uniform ST map 100k edges.

$\mathfrak{q}_{\mathbf{f}}^{\mathbf{a}}$= Unif. tree-decorated map with f faces and a tree of size a.

Why it is interesting to study this family??

Uniform tree-decorated maps

$\mathfrak{q}_{\mathbf{f}}^{\mathbf{a}}$= Unif. tree-decorated map with f faces and a tree of size a.

Why it is interesting to study this family??

- **New statistical mechanic family**

$\mathbb{P}(\mathfrak{q}_f^a = (\mathfrak{m}, \cdot)) \propto \#\{\text{trees of size } a \text{ in } \mathfrak{m}\}$

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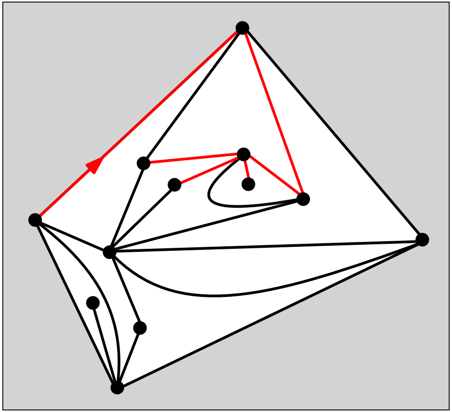
Why it is interesting to study this family??

- **New statistical mechanic family**

$\mathbb{P}(\mathfrak{q}_f^a = (\mathfrak{m}, \cdot)) \propto \#\{\text{trees of size } a \text{ in } \mathfrak{m}\}$

- **It interpolates**
  - $\underline{a = 1}$= Uniform quadrangulations.
  - $\underline{a = f + 1}$= Uniform ST quadrangulations.



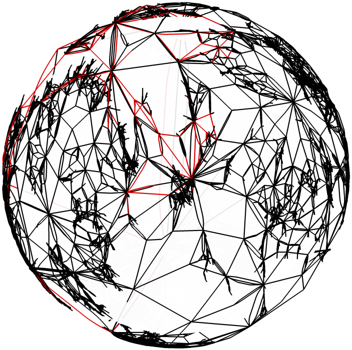
Local limit results

$\mathfrak{q}_{\mathbf{f}}^{\mathbf{a}}$= Unif. tree-decorated map with f faces and a tree of size a.

Theorem (F. & Sepúlveda '19+)

$$\mathfrak{q}_f^a \xrightarrow[local, f \to \infty]{(d)} \mathfrak{q}_\infty^a \xrightarrow[local, a \to \infty]{(d)} \mathfrak{q}_\infty^\infty$$

$\mathfrak{q}_\infty^\infty$ is the "gluing" of $\mathfrak{t}_\infty$ and $UIHPQ^S$.



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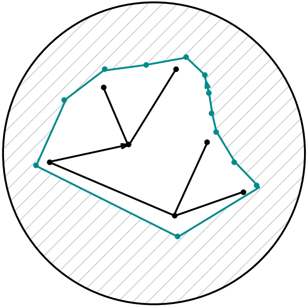

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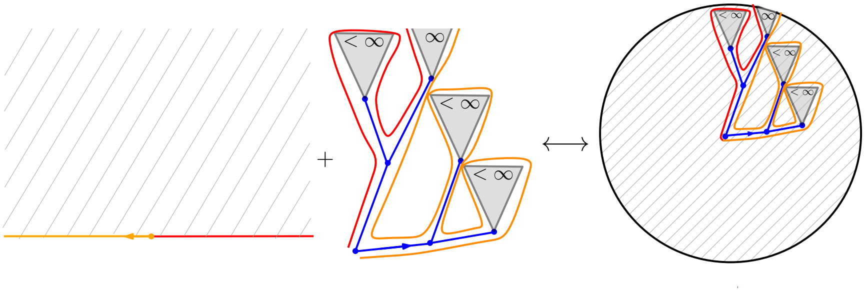


Scaling limit results

$\mathfrak{q}_{\mathbf{f}}^{\mathbf{a}}$= Unif. tree-decorated map with f faces and a tree of size a.

Corollary (F. & Sepúlveda '19+)

Let $\mathfrak{q}_f^{a(f)} = (\mathfrak{q}, \mathfrak{t})$, *with* $a(f) \leq f + 1$. *Then as* $a(f) \to \infty$,

$$\left(\mathfrak{t}, \frac{\mathrm{d}_{\mathrm{Tree}}}{a(f)^{1/2}}\right) \xrightarrow[GH]{(d)} CRT.$$

Scaling limit conjecture

$\mathfrak{q}_{\mathbf{f}}^{\mathbf{a}}$= Unif. tree-decorated map with f faces and a tree of size a.

Conjecture (F. & Sepúlveda '19+)

Let $a(f) = O(f^{\alpha})$. Depending on α as $f \to \infty$

$$\left(\mathfrak{q}_f^{a(f)}, \frac{\mathrm{d}_{\mathrm{map}}}{f^{\beta}}\right) \xrightarrow[GH]{(d)} \begin{cases} \textit{Brownian map} & \text{if } \alpha < 1/2, \beta = 1/4 \textit{(Proved)} \\ \textbf{\textit{Shocked map}} & \text{if } \alpha = 1/2, \beta = 1/4 \textit{(In progress)} \\ \textit{Tree-decorated map} & \text{if } \alpha > 1/2, \\ & \quad \beta = \left(2\chi - \frac{1}{2}\right)\alpha - \chi + \frac{1}{2} \end{cases}$$

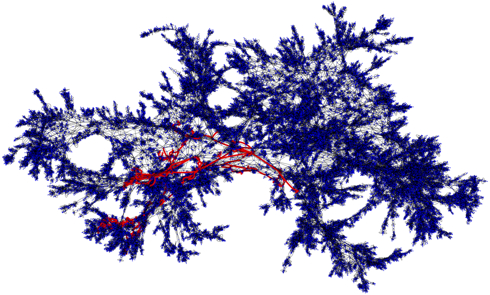
Scaling limit conjecture

q_f^a= Unif. tree-decorated map with f faces and a tree of size a.

Conjecture (F. & Sepúlveda '19+)

Let $a(f) = O(f^\alpha)$. Depending on α as $f \to \infty$

$$\left(q_f^{a(f)}, \frac{d_{map}}{f^\beta}\right) \xrightarrow[GH]{(d)} \begin{cases} \text{Brownian map} & \text{if } \alpha < 1/2, \beta = 1/4 \text{(Proved)} \\ \textbf{Shocked map} & \text{if } \alpha = 1/2, \beta = 1/4 \text{(In progress)} \\ \text{Tree-decorated map} & \text{if } \alpha > 1/2, \\ & \beta = \left(2\chi - \frac{1}{2}\right)\alpha - \chi + \frac{1}{2} \end{cases}$$



Shocked map

Shocked map properties:

- **It is not degenerated** (Proved).
- It should be the gluing of a Brownian disk and a CRT.
- Hausdorff dim. 4 (Proved).
- The tree has Hausdorff dim. 2 (In progress, ≤ 2 proved).
- Homeomorphic to $\mathbb{S}^2$. (Proved).

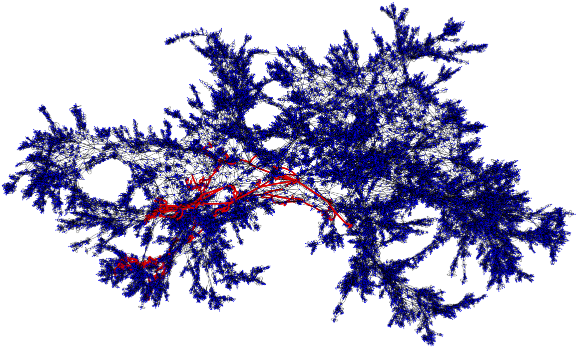


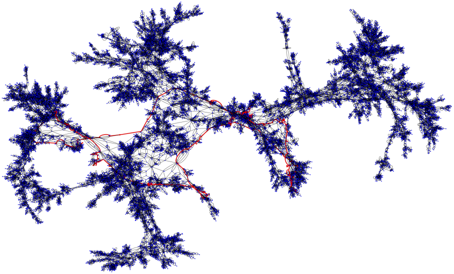
Figure: **Unif. (90k,500) tree-decorated quadrangulation.**

Uniform quadrangulation with a boundary: GH limit

$\mathfrak{q}_{\mathbf{f,p}}$= Unif. quadrangulations with boundary $2p$ and f faces.
For a sequence $(p(f))_{f\in\mathbb{N}}$, define $\bar{p} = \lim p(f)f^{-1/2}$ as $f \to \infty$.

Theorem (Scaling limit (Bettinelli '15))

$$\left(\mathfrak{q}_{f,p(f)}, \frac{\mathrm{d}_{\mathrm{map}}}{s(f,p(f))}\right) \xrightarrow[GH]{(d)} \begin{cases} \textit{Brown. map} & \textit{if } s(f,p(f)) = f^{1/4} \textit{ and } \bar{p} = 0 \\ \textit{Brown. disk} & \textit{if } s(f,p(f)) = f^{1/4} \textit{ and } \bar{p} \in (0,+\infty) \\ \textit{CRT} & \textit{if } s(f,p(f)) = 2p(f)^{1/2} \textit{ and } \bar{p} = \infty \end{cases}$$



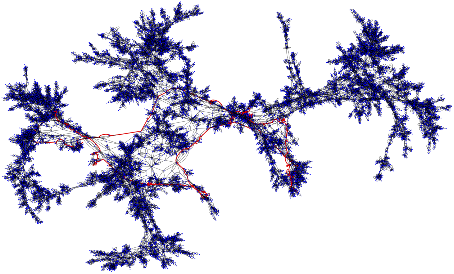
Unif. quad. with 30k interior faces and boundary 173.

Uniform quadrangulation with a simple boundary: GH limit

$\mathfrak{q}_{\mathbf{f},\mathbf{p}}{}^{S}$= Unif. quadrangulations with **simple** boundary $2p$ and f faces.
For a sequence $(p(f))_{f\in\mathbb{N}}$, define $\bar{p} = \lim p(f)f^{-1/2}$ as $f \to \infty$.

Theorem (Scaling limit (Bettinelli, Curien, F., Sepúlveda '19+))

$$\left(\mathfrak{q}_{f,p(f)}{}^{S}, \frac{\mathrm{d}_{\mathrm{map}}}{s(f,p(f))}\right) \xrightarrow[GH]{(d)} \begin{cases} \textit{Brown. map} & \textit{if } s(f,p(f)) = f^{1/4} \textit{ and } \bar{p} = 0 \\ \textit{Brown. disk} & \textit{if } s(f,p(f)) = f^{1/4} \textit{ and } \bar{p} \in (0,+\infty) \\ \textit{CRT} & \textit{if } s(f,p(f)) = 2p(f)^{1/2} \textit{ and } \bar{p} = \infty \end{cases}$$



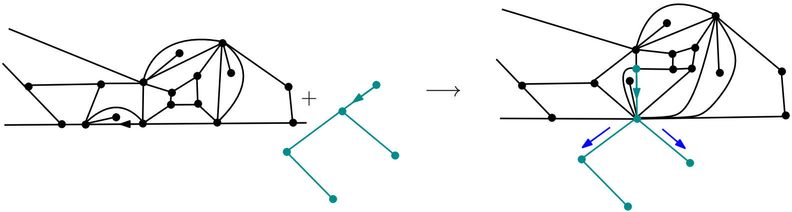
Unif. quad. with 30k interior faces and boundary 173.

Why shocked?

Thanks for your attention!

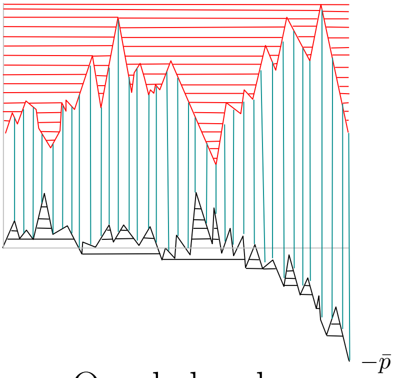
It is not degenerated.

To prove it we do a sequential gluing, tool used to define a peeling.



Then we use the estimates in [Caraceni & Curien, Self-Avoiding Walks on the UIPQ] and the properties of the contour of a tree, to show that distances do not create big shortcuts.

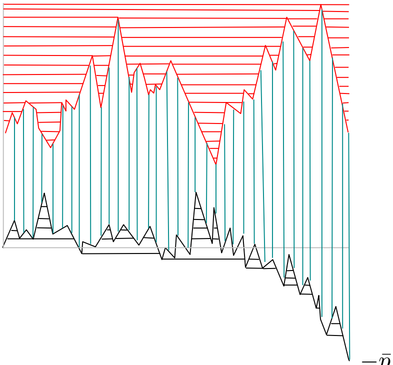
In discrete



Quad. bord p

Homeomorphic to $\mathbb{S}^2$.

In discrete



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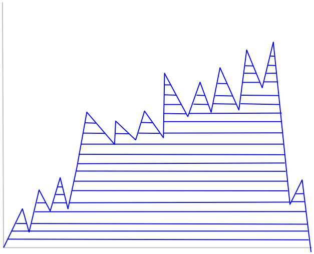
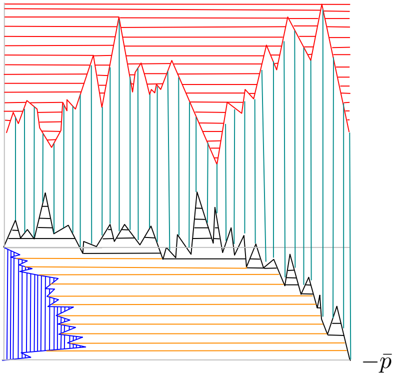


tree

Homeomorphic to $\mathbb{S}^2$.

In discrete



Quad. bord p glued with a tree