Non-bijective scaling limit of maps via restrictions

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(Work with J. Bettinelli, N. Curien and A. Sepúlveda)

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Maps

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Planar map = embedding of a planar graph on the sphere up to homeomorphism.

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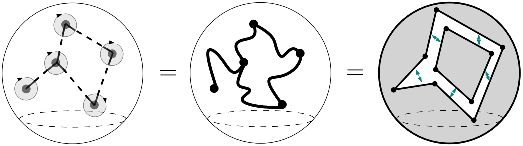

Figure: Tree equivalent ways to define the classes of planar maps.

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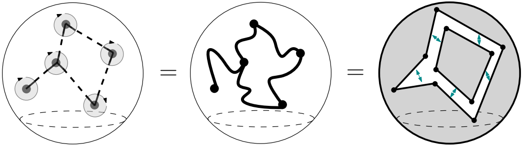
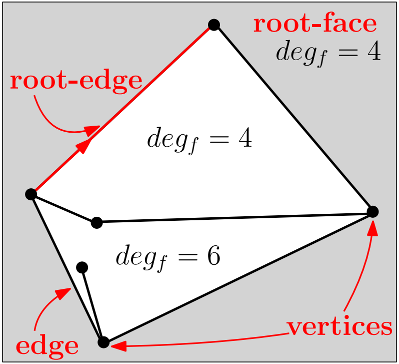


Figure: Tree equivalent ways to define the classes of planar maps.

- A face= A connected component of the complement of the edges.



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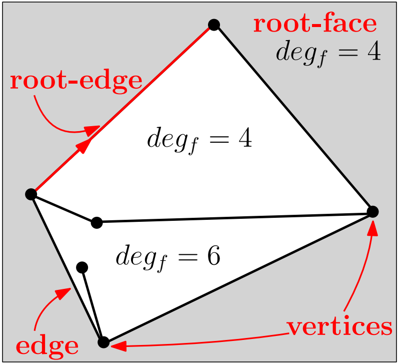

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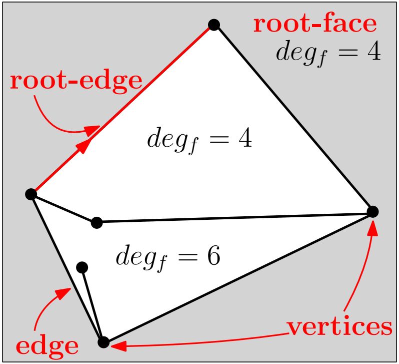

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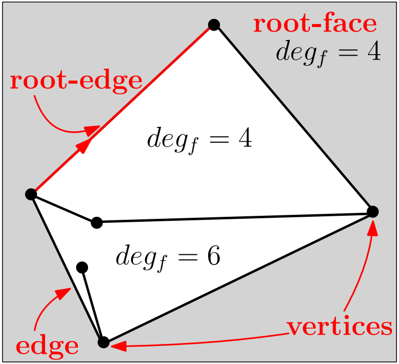
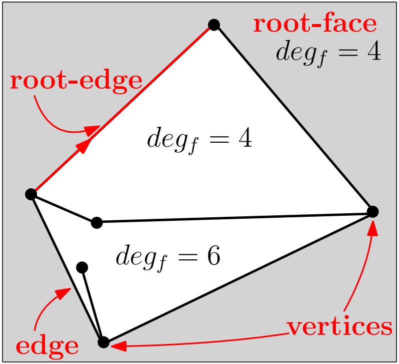


Maps

- A face= A connected component of the complement of the edges.
- The root-edge= distinguished half edge.
- The root-face= face to the left of the root-edge.
- Degree of a face= number of adjacent edges to it.

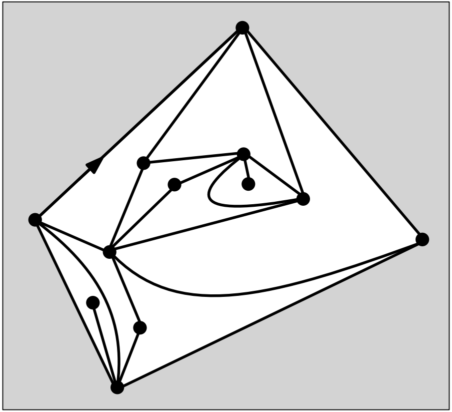


q-angulations

q-angulations: map whose faces have degree q.
Triangulations: 3-angulations.
Quadrangulations: 4-angulations.

These families are in bijection with different families of labelled trees.

[Cori-Vauquelin-Schaeffer '98,
Di-Francesco-Bouttier-Guitter '04].



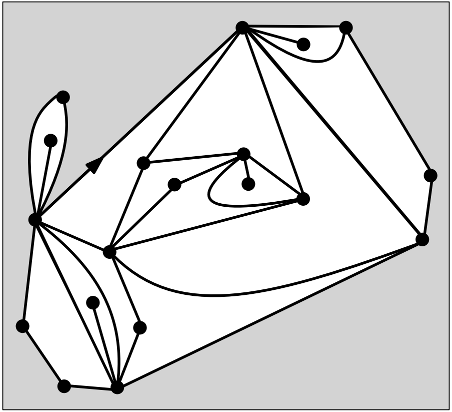
q-angulations with a boundary

q-angulations with a boundary: All faces, but the root-face, have degree q.

Simple boundary: boundary without pinch points.

Some families of maps with a boundary are in bijection with labelled forests.

[Schaeffer '97 ; Poulalhon & Schaeffer '06.; Bettinelli '15]



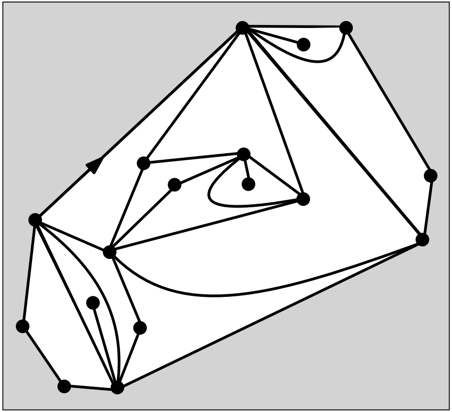
Number of quadrangulations with a simple boundary with:

- f internal faces.
- simple boundary of size $2p$ (root-face of degree $2p$).

$$q_{f,p} \underset{f\to\infty}{\sim} C_p 12^f f^{-5/2}$$

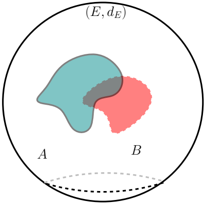
$$C_p \underset{p\to\infty}{\sim} \frac{\sqrt{3p}}{2}\left(\frac{9}{2}\right)^p$$

Analytic [Bouttier & Guitter '09] and bijective [Bernardi & Fusy '17].



Let $(E; d_E)$ be a metric space and $A; B \quad E$. The Hausdor distance is

$$d_H(A; B) = \inf \left\{ \varepsilon > 0 : A \quad B_\varepsilon; B \quad A_\varepsilon \right\}$$



Gromov-Hausdorff topology

Let (E, d_E) be a metric space and $A, B \subset E$. The Hausdorff distance is

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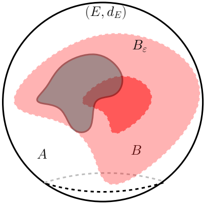
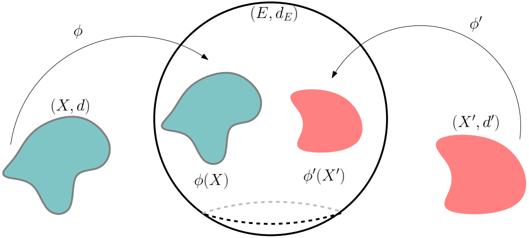


Consider the set S of compact metric spaces up to isometry classes. The Gromov-Hausdorff distance between two metric spaces (X, d) and (X', d') is defined as

$$\mathrm{d_{GH}}((\mathrm{X}, \mathrm{d}), (\mathrm{X}', \mathrm{d}')) = \inf \mathrm{d_H}(\varphi(\mathrm{X}), \varphi'(\mathrm{X}'))$$

where the infimum is taken over all metric spaces $(E, \mathrm{d_E})$ and all isometric embeddings φ, φ' from X, X' respectively into E.



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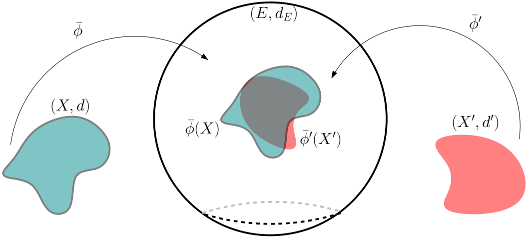

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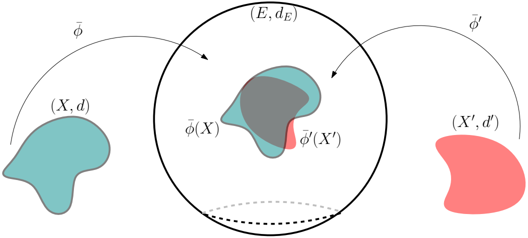

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Proposition

The function $\mathrm{d_{GH}}$ *induces a metric on* S*. The space* $(S, \mathrm{d_{GH}})$ *is separable and complete.*

Uniform quadrangulations

q_f= Unif. quadrangulation with f faces.

Theorem (Le Gall '13, Miermont '13)

$$\left(q_f, \frac{d_{map}}{f^{1/4}}\right) \xrightarrow[GH]{(d)} \textit{Brownian map}$$

Properties

- *Hausdor dim. is 4 (Le Gall '07).*
- *Homeomorphic to S^2 (Le Gall & Paulin '08).*

Unif. quadrangulation 30k faces.

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Universal: q-angulations

- $q \geq 3$ (Le Gall '13, Miermont '13, Addario-Berry & Albenque '19).

Others

- $q = 3, 4$ simple (Addario-Berry & Albenque '13).
- Bipartite maps with prescribed face degree sequence (Marzouk '19).

Unif. quadrangulation 30k faces.

$q_{f,p}$ = Unif. quadrangulations with a boundary of size $2p$ and f faces. For a sequence $(p(f))_{f\in\mathbb{N}}$, define $\bar{p} = \lim p(f) f^{-1/2}$ as $f \to \infty$.

Theorem (Scaling limit (Bettinelli '15))

$$\left(q_{f,p(f)}, \frac{d_{map}}{s(f,p(f))}\right) \xrightarrow[GH]{(d)} \begin{cases} \text{Brownian map} & \text{if } s(f,p(f)) = f^{1/4} \text{ and } \bar{p} = 0 \\ \text{Brownian disk} & \text{if } s(f,p(f)) = f^{1/4} \text{ and } \bar{p} \in (0,+\infty) \\ \text{CRT} & \text{if } s(f,p(f)) = (2p(f))^{1/2} \text{ and } \bar{p} = \infty \end{cases}$$

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Properties (Bettinelli & Miermont '15)

Brownian disk properties

- The boundary is simple.
- Hausdorff dim. 4 in the interior, 2 in the boundary.
- Homeomorphic to the disk $\mathbb{D}$.

Unif. quad. with 30k interior faces and boundary 173.

Same trichotomy

- Bipartite maps with prescribed face degree sequence (Marzouk '19).

Critical case (boundary $\sqrt{\text{volume}}$)

- Triangulations ($q = 3$) with simple boundary (Albenque, Hölden & Sun '19).

Unif. quad. with 30k interior faces and boundary 173.

These results are all obtained from bijections !

Our main result is a non-bijective technique that allows us to obtain the convergence of a given model of random maps using a reference converging model of random maps.

Consider a collection of i.i.d. r.v. $\{X_i\}_{i \in \mathbb{N}}$ With $P(X_1 = 1) = P(X_1 = -1) = 1/2$. Denote by $S_n = X_1 + X_2 + \dots + X_n$ and set $S_0 = 0$.

How to prove that in $C([0,1])$

$$(Y_t)_{t \in [0,1]} := \left(\frac{S_{Nt}}{\sqrt{N}} \,\middle|\, S_N = 0\right)_{t \in [0,1]} \xrightarrow{(d)} \left(Br^b_t\right)_{t \in [0,1]} ? \qquad \text{(Model)}$$

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Suppose given that for $\varepsilon \to 0$ as $N \to \infty$

$$(\overline{Y}_t)_{t\in[0,1]} := \left(\frac{S_{Nt}}{\sqrt{N}} \,\Big|\, S_n \in [-\varepsilon\sqrt{N}, \varepsilon\sqrt{N}]\right)_{t\in[0,1]} \xrightarrow{(d)} \left(Br^b_t\right)_{t\in[0,1]}$$

(Reference)

We will compare this two processes.

Figure: In green the reference $(\overline{Y}_t)$ and in blue the model (Y_t) (bridge).

Figure: In green the reference $(\overline{Y}_t)$ and in blue the model (Y_t) (bridge), before scaling.

Consider

- Model : $(P_n : n \in \mathbb{N})$ supported in the set of maps.
- Reference: $(\overline{P}_n : n \in \mathbb{N})$ supported in the set of maps

And two sets of restrictions

- $\{R_n^{1,\varepsilon} : n \in \mathbb{N}, \varepsilon > 0\}$.
- $\{R_n^{2,\varepsilon} : n \in \mathbb{N}, \varepsilon > 0\}$.

De ne the following conditions

C1) Convergent reference $\overline{P}_n$! $\overline{P}$ in the GH topology.

C2) The Radon-Nikodym derivative between the restrictions in the reference and in the model is 1+ o(").

C3) The complement of both restrictions are disjoint and small with probability 1 o(").

Define the following conditions

C1) Convergent reference $\overline{P}_n \to \overline{P}$ in the GH topology.

C2) The Radon-Nikodym derivative between the restrictions in the reference and in the model is $1 + o(\varepsilon)$.

C3) The complement of both restrictions are disjoint and "small" with probability $1 - o(\varepsilon)$.

Theorem (Bettinelli, Curien, F., Sepúlveda '19+)

If (C1- C3) are satisfied, then

$$P_n \to \overline{P} \quad \text{as } n \to \infty$$

in the Gromov-Hausdorff topology

Figure: Two restrictions

Tightness: (C1) & (C2) establish tightness of each restriction, which is enough to obtain the complete tightness as soon as the union of both restrictions covers the whole map (C3) .

Fin. dim. dist: One restriction su ces as soon as the complement of restrictions are small (C3) .

Limit: The same limit as the reference (C1)

$q^{S}_{f,p}$ = Unif. quadrangulations with simple boundary $2p$ and f faces.
For a sequence $(p(f))_{f \in \mathbb{N}}$, define $p = \lim p(f) f^{-1/2}$ as $f \to \infty$.

Theorem (Bettinelli, Curien, F., Sepúlveda '19+)

If $p \in (0; +\infty)$

$$\left(q^{S}_{f;p(f)}; \frac{d_{map}}{f^{1/4}}\right) \xrightarrow[GH]{(d)} \text{Brown. disk}$$

Unif. quad. with 30k interior faces and boundary 173.

Recall that a.s. the boundary of the Brownian disk is simple.

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We will extract from a quad. with general boundary m a simple boundary quadrangulation called the core ($\mathrm{Core}(m)$).

Figure: Quad. with general boundary.

Figure: Core decomposition.

Since the core is a pruning of a general boundary quadrangulation, it has random volume and random boundary size. It concentrates

Proposition (Gwynne & Miller '16))

The core obtained from $q_{n;q_n}$ satisfies the following.

- Conditionally on its perimeter p_n and its area a_n, it is a uniform quad. with simple boundary.

$$a_n/n \to 1 \quad \text{and} \quad p_n/q_n \to 1/3$$

- In law when $p \in (0, 1)$

To apply our theorem we use

- Model: Uniform random quad. with a simple boundary (n faces, p_n boundary).
- Reference: Core of a uniform quad. with general boundary (n faces, $3p_n$ boundary).

The core is a reference satisfying (C1)

Theorem (Gwynne & Miller '16)

If $p \in (0, +\infty)$

$$\left(\mathrm{Core}(q_{n,3p_n}), \frac{d_{map}}{n^{1/4}}\right) \xrightarrow[GH]{(d)} \text{Brown. disk}$$

Figure: Ball growing hitting time.

Figure: Restriction.

We do the same procedure both in the Model and the Reference.

Recall that

$$q_{n,p} \underset{n\to\infty}{\sim} C_p 12^n n^{-5/2}$$

In both the model and the reference the probability to see a restriction s_0 with area a

$$\mathbb{P}(R_n^{i,\varepsilon}(S_{n,p}) = s_0) = \frac{\#\text{ Ways to complete the holes } s_1}{q_{n,p}}$$

$$= \frac{q_{n-a,\ell_1+\ell_2}}{q_{n,p}}$$

Using the countings, in the same fashion as for walks, we obtain that the Radon-Nikodym derivative for each restriction satisfies

$$\frac{d\overline{P}_N^{i,\varepsilon}}{dP_N^{i,\varepsilon}}(s_0) = 1 + o(\varepsilon).$$

So condition C2) is satisfied.

It remains to check condition C3).

- The complement of the restrictions are disjoint w.h.p. for the chosen points:
  - In the continuum when $\varepsilon \to 0$, the complement of restrictions shrink to a point.
- The complement of the two restrictions are small w.h.p.:
  - In the critical case, the zones that remain after taking the limit are zones where the boundary are of order $\sqrt{\text{volume}}$, because of this and the continuity of the intersection points, the complement of restrictions are "small".

Final comments

With a reference, counting results and a kind of "Markovian" property, one can obtain scaling limits.
Other possible results by applying this theorem:

- Other simple boundary objects as bipartite maps with prescribed face degree sequence.
- Simple maps (without loops or multiple edges) by extracting the 2-connected core.
- Maps with more than one simple hole, by treating one by one each hole.
- The precedents in higher genus.