## Non-bijective scaling limit of maps via restrictions

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## Maps



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Planar map $=$ embedding of a planar graph on the sphere up to homeomorphism.

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- A face $=\mathrm{A}$ connected component of the complement of the edges.
- The root-edge $=$ distinguished half edge.
- The root-face= face to the left of the root-edge.
- Degree of a face $=$ number of adjacent edges to it.



## $q$-angulations

$q$-angulations: map whose faces have degree $q$.
Triangulations: 3-angulations. Quadrangulations: 4-angulations.

These families are in bijection with different families of labelled trees.
[Cori-Vauquelin-Schaeffer '98, Di-Francesco-Bouttier-Guitter '04].


## $q$-angulations with a boundary

$q$-angulations with a boundary: All faces, but the root-face, have degree $q$.

Simple boundary: boundary without pinch points.

Some families of maps with a boundary are in bijection with labelled forests.
[Schaeffer '97 ; Poulalhon \& Schaeffer '06.; Bettinelli '15]


## Quadrangulations with a simple boundary

Number of quadrangulations with a simple boundary with:

- $f$ internal faces.
- simple boundary of size $2 p$ (root-face of degree $2 p$ ).

$$
\begin{gathered}
q_{f, p} \underset{f \rightarrow \infty}{\sim} C_{p} 12^{f} f^{-5 / 2} \\
C_{p} \underset{p \rightarrow \infty}{\sim} \frac{\sqrt{3 p}}{2 \pi}\left(\frac{9}{2}\right)^{p}
\end{gathered}
$$

Analytic [Bouttier \& Guitter '09] and bijective [ Bernardi \& Fusy '17 ].


## Gromov-Hausdorff topology

Let $\left(E, d_{E}\right)$ be a metric space and $A, B \subset E$. The Hausdorff distance is $d_{H}(A, B)=\inf \left\{\varepsilon>0: A \subset B_{\varepsilon}, B \subset A_{\varepsilon}\right\}$


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## Gromov-Hausdorff topology

Consider the set $S$ of compact metric spaces up to isometry classes. The Gromov-Hausdorff distance between two metric spaces $(X, d)$ and $\left(X^{\prime}, d^{\prime}\right)$ is defined as

$$
\mathrm{d}_{\mathrm{GH}}\left((\mathrm{X}, \mathrm{~d}),\left(\mathrm{X}^{\prime}, \mathrm{d}^{\prime}\right)\right)=\inf \mathrm{d}_{\mathrm{H}}\left(\phi(\mathrm{X}), \phi^{\prime}\left(\mathrm{X}^{\prime}\right)\right)
$$

where the infimum is taken over all metric spaces $\left(E, \mathrm{~d}_{\mathrm{E}}\right)$ and all isometric embeddings $\phi, \phi^{\prime}$ from $X, X^{\prime}$ respectively into $E$.

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## Proposition

The function $\mathrm{d}_{\mathrm{GH}}$ induces a metric on $S$. The space $\left(S, \mathrm{~d}_{\mathrm{GH}}\right)$ is separable and complete.

## Uniform quadrangulations

$\mathfrak{q}_{\mathfrak{f}}=$ Unif. quadrangulation with $f$ faces.
Theorem (Le Gall '13, Miermont '13)

$$
\left(\mathfrak{q}_{f}, \frac{d_{\text {map }}}{f^{1 / 4}}\right) \xrightarrow[G H]{(d)} \text { Brownian map }
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## Properties

- Hausdorff dim. is 4 (Le Gall '07).
- Homeomorphic to $\mathbb{S}^{2}$ (Le Gall \& Paulin '08).



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## Universal: $q$-angulations

- $q \geq 3$ (Le Gall ' 13 , Miermont ' 13 , Addario-Berry \& Albenque '19).


## Others

- $q=3,4$ simple (Addario-Berry \& Albenque '13).
- Bipartite maps with prescribed face degree sequence (Marzouk '19).



## Uniform quadrangulation with a boundary: GH limit

$\mathfrak{q}_{\mathfrak{f}, \mathbf{p}}=$ Unif. quadrangulations with a boundary of size $2 p$ and $f$ faces. For a sequence $(p(f))_{f \in \mathbb{N}}$, define $\bar{p}=\lim p(f) f^{-1 / 2}$ as $f \rightarrow \infty$.

## Theorem (Scaling limit (Bettinelli '15))

$$
\left(\mathfrak{q}_{f, p(f)}, \frac{\mathrm{d}_{\text {map }}}{s(f, p(f))}\right) \xrightarrow[G H]{(d)} \begin{cases}\text { Brownian map } & \text { if } s(f, p(f))=f^{1 / 4} \text { and } \bar{p}=0 \\ \text { Brownian disk } & \text { if } s(f, p(f))=f^{1 / 4} \text { and } \bar{p} \in(0,+\infty) \\ C R T & \text { if } s(f, p(f))=2 p(f)^{1 / 2} \text { and } \bar{p}=\infty\end{cases}
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## Properties (Bettinelli \& Miermont

 '15)Brownian disk properties

- The boundary is simple.
- Hausdorff dim. 4 in the interior, 2 in the boundary.
- Homeomorphic to the disk $2 d$.


Unif. quad. with 30k interior faces and boundary 173.

## Universal behavior

"Same" trichotomy

- Bipartite maps with prescribed face degree sequence (Marzouk '19).

Critical case (boundary $\approx \sqrt{\text { volume }}$ )

- Triangulations $(q=3)$ with simple boundary (Albenque, Hölden \& Sun '19).


Unif. quad. with 30k interior faces and boundary 173.

These results are all obtained from bijections!

Our main result is a non-bijective technique that allows us to obtain the convergence of a given model of random maps using a reference converging model of random maps.

## Related example

Consider a collection of i.i.d. r.v. $\left\{X_{i}\right\}_{i \in \mathbb{N}}$ With $\mathbb{P}\left(X_{1}=1\right)=\mathbb{P}\left(X_{1}=-1\right)=1 / 2$. Denote by $S_{n}=X_{1}+X_{2}+\ldots+X_{n}$ and set $S_{0}=0$.

How to prove that in $C([0,1])$

$$
\begin{equation*}
\left(Y_{t}\right)_{t \in[0,1]}:=\left(\left.\frac{S_{N t}}{\sqrt{N}} \right\rvert\, S_{N}=0\right)_{t \in[0,1]} \xrightarrow{(d)}\left(B r_{t}^{b}\right)_{t \in[0,1]} ? \tag{Model}
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Suppose given that for $\varepsilon \rightarrow 0$ as $N \rightarrow \infty$

$$
\left(\bar{Y}_{t}\right)_{t \in[0,1]}:=\left(\left.\frac{S_{N t}}{\sqrt{N}} \right\rvert\, S_{n} \in[-\varepsilon \sqrt{N}, \varepsilon \sqrt{N}]\right)_{t \in[0,1]} \xrightarrow{(d)}\left(\operatorname{Br}_{t}^{b}\right)_{t \in[0,1]}
$$

(Reference)
We will compare this two processes.


Figure: In green the reference $\left(\bar{Y}_{t}\right)$ and in blue the model $\left(Y_{t}\right)$ (bridge).


Figure: In green the reference $\left(\bar{Y}_{t}\right)$ and in blue the model $\left(Y_{t}\right)$ (bridge), before scaling.

## Random maps setting

Consider

- Model: $\left(\mathbb{P}_{n}: n \in \mathbb{N}\right)$ supported in the set of maps.
- Reference: $\left(\overline{\mathbb{P}}_{n}: n \in \mathbb{N}\right)$ supported in the set of maps.

And two sets of restrictions

- $\left\{R_{n}^{1, \varepsilon}: n \in \mathbb{N}, \varepsilon>0\right\}$.
- $\left\{R_{n}^{2, \varepsilon}: n \in \mathbb{N}, \varepsilon>0\right\}$.

Define the following conditions
C1) Convergent reference $\overline{\mathbb{P}}_{n} \rightarrow \overline{\mathbb{P}}$ in the GH topology.
C2) The Radon-Nikodym derivative between the restrictions in the reference and in the model is $1+o(\varepsilon)$.
C3) The complement of both restrictions are disjoint and "small" with probability $1-o(\varepsilon)$.

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## Theorem (Bettinelli, Curien, F., Sepúlveda '19+)

If (C1- C3) are satisfied, then

$$
\mathbb{P}_{n} \rightarrow \overline{\mathbb{P}} \quad \text { as } n \rightarrow \infty
$$

in the Gromov-Hausdorff topology

## Idea of the proof



Figure: Two restrictions

- Tightness: (C1) \& (C2) establish tightness of each restriction, which is enough to obtain the complete tightness as soon as the union of both restrictions covers the whole map (C3).
- Fin. dim. dist: One restriction suffices as soon as the complement of restrictions are small (C3).
- Limit: The same limit as the reference (C1)


## Application: GH limit of simple boundary quad.

$\mathfrak{q f}_{\mathrm{f}, \mathrm{p}}{ }^{S}=$ Unif. quadrangulations with simple boundary $2 p$ and $f$ faces.
For a sequence $(p(f))_{f \in \mathbb{N}}$, define $\bar{p}=\lim p(f) f^{-1 / 2}$ as $f \rightarrow \infty$.

## Theorem (Bettinelli, Curien, F., Sepúlveda '19+)

If $\bar{p} \in(0,+\infty)$

$$
\left(\mathfrak{q}_{f, p(f)}{ }^{s}, \frac{d_{\text {map }}}{f^{1 / 4}}\right) \xrightarrow[G H]{(d)} \text { Brown. disk }
$$



Unif. quad. with 30k interior faces and boundary 173.

Recall that a.s. the boundary of the Brownian disk is simple.

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We will extract from a quad. with general boundary $\mathfrak{m}$ a simple boundary quadrangulation $\mathfrak{s m}$ called the core (Core(m)).


Figure: Quad. with general boundary.


Figure: Core decomposition.

Since the core is a pruning of a general boundary quadrangulation, it has random volume and random boundary size. It concentrates

## Proposition (Gwynne \& Miller '16))

The core obtained from $\mathfrak{q}_{n, q_{n}}$ satisfies the following.

- Conditionally on its perimeter $p_{n}$ and its area $a_{n}$, it is a uniform quad. with simple boundary.

$$
a_{n} / n \rightarrow 1 \quad \text { and } p_{n} / q_{n} \rightarrow 1 / 3
$$

In law when $\bar{p} \in(0, \infty)$
To apply our theorem we use

- Model: Uniform random quad. with a simple boundary ( $n$ faces, $p_{n}$ boundary).
- Reference: Core of a uniform quad. with general boundary ( $n$ faces, $3 p_{n}$ boundary).

The core is a reference satisfying (C1)
Theorem (Gwynne \& Miller '16)
If $\bar{p} \in(0,+\infty)$

$$
\left(\operatorname{Core}\left(\mathfrak{q}_{n, 3 p_{n}}\right), \frac{d_{\text {map }}}{n^{1 / 4}}\right) \xrightarrow[G H]{(d)} \text { Brown. disk }
$$

## Restrictions



Figure: Ball growing hitting time.


Figure: Restriction.

We do the same procedure both in the Model and the Reference.

## Radon-Nikodym

Recall that
$q_{n, p} \underset{n \rightarrow \infty}{\sim} C_{p} 12^{n} n^{-5 / 2}$


In both the model and the reference the probability to see a restriction $\mathfrak{s}_{0}$ with area a

$$
\begin{aligned}
\mathbb{P}\left(R_{n}^{i, \varepsilon}\left(S_{n, p}\right)=\mathfrak{s}_{0}\right) & =\frac{\# \text { Ways to complete the hole } \mathfrak{s}_{1}}{q_{n, p}} \\
& =\frac{q_{n-a, \ell_{1}+\ell_{2}}}{q_{n, p}}
\end{aligned}
$$

Using the countings, in the same fashion as for walks, we obtain that the Radon-Nikodym derivative for each restriction satisfies

$$
\frac{d \overline{\mathbb{P}}_{N}^{i, \varepsilon}}{d \mathbb{P}_{N}^{i, \varepsilon}}\left(\mathfrak{s}_{0}\right)=1+o(\varepsilon)
$$

So condition C2) is satisfied.
It remains to check condition C3).

- The complement of the restrictions are disjoint w.h.p. for the chosen points:

In the continuum when $\varepsilon \rightarrow 0$, the complement of restrictions shrink to a point.

- The complement of the two restrictions are small w.h.p.:

In the critical case, the zones that remain after taking the limit are zones where the boundary are of order $\sqrt{\text { volume }}$, because of this and the continuity of the intersection points, the complement of restrictions are "small".

With a reference, counting results and a kind of "Markovian" property, one can obtain scaling limits.
Other possible results by applying this theorem:

- Other simple boundary objects as bipartite maps with prescribed face degree sequence.
- Simple maps (without loops or multiple edges ) by extracting the 2-connected core.
- Maps with more than one simple hole, by treating one by one each hole.
- The precedents in higher genus.

