# Non-bijective scaling limit of maps via restrictions

#### Luis Fredes (Work with J. Bettinelli, N. Curien and A. Sepúlveda)

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**Planar map** = embedding of a planar graph on the sphere up to homeomorphism.



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- The **root-face**= face to the left of the root-edge.
- Degree of a face= number of adjacent edges to it.



*q*-angulations: map whose faces have degree *q*. Triangulations: 3-angulations. Quadrangulations: 4-angulations.

These families are in bijection with different families of labelled trees.

[Cori-Vauquelin-Schaeffer '98, Di-Francesco-Bouttier-Guitter '04].



q-angulations with a boundary: All faces, but the root-face, have degree q.

Simple boundary: boundary without pinch points.

Some families of maps with a boundary are in bijection with labelled forests.

[Schaeffer '97 ; Poulalhon & Schaeffer '06.; Bettinelli '15]



Number of quadrangulations with a simple boundary with:

- f internal faces.
- **simple boundary** of size 2*p* (root-face of degree 2*p*).

$$q_{f,p} \underset{f \to \infty}{\sim} C_p 12^f f^{-5/2}$$
$$C_p \underset{p \to \infty}{\sim} \frac{\sqrt{3p}}{2\pi} \left(\frac{9}{2}\right)^p$$

Analytic [Bouttier & Guitter '09] and bijective [ Bernardi & Fusy '17 ].



Let  $(E, d_E)$  be a metric space and  $A, B \subset E$ . The **Hausdorff distance** is

$$\mathsf{d}_{\mathsf{H}}(\mathsf{A},\mathsf{B}) = \mathsf{inf}\left\{\varepsilon > \mathsf{0}: \mathsf{A} \subset \mathsf{B}_{\varepsilon}, \mathsf{B} \subset \mathsf{A}_{\varepsilon}\right\}$$



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Consider the set S of compact metric spaces up to isometry classes. The **Gromov-Hausdorff distance** between two metric spaces (X, d) and (X', d') is defined as

$$\mathsf{d}_{\mathsf{GH}}((\mathsf{X},\mathsf{d}),(\mathsf{X}',\mathsf{d}')) = \inf \mathsf{d}_{\mathsf{H}}(\phi(\mathsf{X}),\phi'(\mathsf{X}'))$$

where the infimum is taken over all metric spaces  $(E, d_E)$  and all isometric embeddings  $\phi, \phi'$  from X, X' respectively into E.



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#### Proposition

The function  $d_{\mathsf{GH}}$  induces a metric on S. The space  $(S,d_{\mathsf{GH}})$  is separable and complete.

# Uniform quadrangulations

 $q_f = Unif.$  quadrangulation with f faces.

Theorem (Le Gall '13, Miermont '13)

$$\left(\mathfrak{q}_{f}, \frac{\mathsf{d}_{\mathsf{map}}}{f^{1/4}}\right) \xrightarrow[GH]{(d)} Brownian map$$

#### Properties

- Hausdorff dim. is 4 (Le Gall '07).
- Homeomorphic to S<sup>2</sup> (Le Gall & Paulin '08).



Unif. quadrangulation 30k faces.

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## Universal: q-angulations

 q ≥ 3 (Le Gall '13, Miermont '13, Addario-Berry & Albenque '19).

#### Others

- q = 3,4 simple (Addario-Berry & Albenque '13).
- Bipartite maps with prescribed face degree sequence (Marzouk '19).



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# Uniform quadrangulation with a boundary: GH limit

 $\mathfrak{q}_{f,p}$ = Unif. quadrangulations with a boundary of size 2p and f faces. For a sequence  $(p(f))_{f\in\mathbb{N}}$ , define  $\overline{p} = \lim p(f)f^{-1/2}$  as  $f \to \infty$ .

#### Theorem (Scaling limit (Bettinelli '15))

$$\begin{pmatrix} \mathfrak{q}_{f,p(f)}, \frac{\mathsf{d}_{\mathsf{map}}}{s(f,p(f))} \end{pmatrix} \xrightarrow{(d)}_{GH} \begin{cases} \text{Brownian map} & \text{if } s(f,p(f)) = f^{1/4} \text{ and } \overline{p} = 0 \\ \text{Brownian disk} & \text{if } s(f,p(f)) = f^{1/4} \text{ and } \overline{p} \in (0,+\infty) \\ CRT & \text{if } s(f,p(f)) = 2p(f)^{1/2} \text{ and } \overline{p} = \infty \end{cases}$$

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## Properties (Bettinelli & Miermont '15)

Brownian disk properties

- The boundary is simple.
- Hausdorff dim. 4 in the interior, 2 in the boundary.
- Homeomorphic to the disk 2d.





## Universal behavior

"Same" trichotomy

• Bipartite maps with prescribed face degree sequence (Marzouk '19). Critical case (boundary  $\approx \sqrt{volume}$ )

• Triangulations (q = 3) with simple boundary (Albenque, Hölden & Sun '19).



Unif. quad. with 30k interior faces and boundary 173.

These results are all obtained from **bijections**!

Our main result is a **non-bijective** technique that allows us to obtain the convergence of a given **model** of random maps using a **reference** converging model of random maps.

## Related example

Consider a collection of i.i.d. r.v.  $\{X_i\}_{i\in\mathbb{N}}$  With  $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$ . Denote by  $S_n = X_1 + X_2 + \ldots + X_n$  and set  $S_0 = 0$ .

How to prove that in C([0, 1])

$$(Y_t)_{t\in[0,1]} := \left(\frac{S_{Nt}}{\sqrt{N}} \left| S_N = 0 \right)_{t\in[0,1]} \xrightarrow{(d)} \left( Br_t^b \right)_{t\in[0,1]}?$$
 (Model)

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Suppose given that for  $\varepsilon \to 0$  as  $\textit{N} \to \infty$ 

$$(\overline{Y}_t)_{t\in[0,1]} := \left(\frac{S_{Nt}}{\sqrt{N}} \left| S_n \in [-\varepsilon\sqrt{N}, \varepsilon\sqrt{N}] \right)_{t\in[0,1]} \xrightarrow{(d)} (Br_t^b)_{t\in[0,1]} \right|$$
(Reference)

We will compare this two processes.



Figure: In green the reference  $(\overline{Y}_t)$  and in blue the model  $(Y_t)$  (bridge).



Figure: In green the reference  $(\overline{Y}_t)$  and in blue the model  $(Y_t)$  (bridge), before scaling.

# Consider

- Model:  $(\mathbb{P}_n : n \in \mathbb{N})$  supported in the set of maps.
- **Reference**:  $(\overline{\mathbb{P}}_n : n \in \mathbb{N})$  supported in the set of maps.

And two sets of restrictions

• {
$$R_n^{1,\varepsilon}$$
 :  $n \in \mathbb{N}, \varepsilon > 0$ }.  
• { $R_n^{2,\varepsilon}$  :  $n \in \mathbb{N}, \varepsilon > 0$ }.

Define the following conditions

- C1) Convergent **reference**  $\overline{\mathbb{P}}_n \to \overline{\mathbb{P}}$  in the GH topology.
- C2) The Radon-Nikodym derivative between the restrictions in the **reference** and in the **model** is  $1 + o(\varepsilon)$ .
- C3) The complement of both restrictions are disjoint and "small" with probability  $1 o(\varepsilon)$ .

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#### Theorem (Bettinelli, Curien, F., Sepúlveda '19+)

If (C1- C3) are satisfied, then

$$\mathbb{P}_n o \overline{\mathbb{P}}$$
 as  $n \to \infty$ 

in the Gromov-Hausdorff topology

# Idea of the proof



Figure: Two restrictions

- Tightness: (C1) & (C2) establish tightness of each restriction, which is enough to obtain the complete tightness as soon as the union of both restrictions covers the whole map (C3).
- Fin. dim. dist: One restriction suffices as soon as the complement of restrictions are small (C3).
- Limit: The same limit as the reference (C1)

# Application: GH limit of simple boundary quad.

 $q_{f,p}{}^{S} = \text{Unif. quadrangulations with simple boundary } 2p \text{ and } f \text{ faces.}$ For a sequence  $(p(f))_{f \in \mathbb{N}}$ , define  $\overline{p} = \lim p(f)f^{-1/2}$  as  $f \to \infty$ .

#### Theorem (Bettinelli, Curien, F., Sepúlveda '19+)

If  $\bar{p} \in (0, +\infty)$ 

$$\left(\mathfrak{q}_{f,p(f)}{}^{\mathsf{S}}, \frac{\mathsf{d}_{\mathsf{map}}}{f^{1/4}}\right) \xrightarrow[\mathsf{GH}]{(d)} Brown. disk$$



Unif. quad. with 30k interior faces and boundary 173.

Recall that a.s. the boundary of the Brownian disk is simple.

# Proof idea

Recall that a.s. the boundary of the Brownian disk is **simple**. We will extract from a quad. with general boundary  $\mathfrak{m}$  a simple boundary quadrangulation  $\mathfrak{sm}$  called the **core** (*Core*( $\mathfrak{m}$ )).



Figure: Quad. with general boundary.



Figure: Core decomposition.

Since the **core** is a pruning of a general boundary quadrangulation, it has random volume and random boundary size. It concentrates

#### Proposition (Gwynne & Miller '16))

The **core** obtained from  $q_{n,q_n}$  satisfies the following.

• Conditionally on its perimeter *p<sub>n</sub>* and its area *a<sub>n</sub>*, it is a uniform quad. with simple boundary.

۲

$$a_n/n 
ightarrow 1$$
 and  $p_n/q_n 
ightarrow 1/3$ 

In law when  $\bar{p} \in (0,\infty)$ 

To apply our theorem we use

- **Model**: Uniform random quad. with a simple boundary (*n* faces, *p<sub>n</sub>* boundary).
- **Reference**: Core of a uniform quad. with general boundary (*n* faces, 3*p<sub>n</sub>* boundary).

#### The core is a reference satisfying (C1)

# Theorem (Gwynne & Miller '16) If $\bar{p} \in (0, +\infty)$ $\left( Core(q_{n,3p_n}), \frac{d_{map}}{n^{1/4}} \right) \xrightarrow{(d)}{GH} Brown. disk$





Figure: Ball growing hitting time.

Figure: Restriction.

We do the same procedure both in the Model and the Reference.

# Radon-Nikodym

Recall that



In both the model and the reference the probability to see a restriction  $\mathfrak{s}_0$  with area a

$$\mathbb{P}(R_n^{i,\varepsilon}(S_{n,p}) = \mathfrak{s}_0) = \frac{\#\mathsf{Ways to complete the hole }\mathfrak{s}_1}{q_{n,p}}$$
$$= \frac{q_{n-a,\ell_1+\ell_2}}{q_{n,p}}$$

Using the countings, in the same fashion as for walks, we obtain that the Radon-Nikodym derivative for each restriction satisfies

$$rac{d\overline{\mathbb{P}}_{N}^{i,arepsilon}}{d\mathbb{P}_{N}^{i,arepsilon}}(\mathfrak{s}_{0})=1+o(arepsilon).$$

So condition C2) is satisfied.

It remains to check condition C3).

- The complement of the restrictions are disjoint w.h.p. for the chosen points: In the continuum when  $\varepsilon \to 0$ , the complement of restrictions shrink to a point.
- The complement of the two restrictions are small w.h.p.:

In the critical case, the zones that remain after taking the limit are zones where the boundary are of order  $\sqrt{volume}$ , because of this and the continuity of the intersection points, the complement of restrictions are "small".

With a **reference**, counting results and a kind of "Markovian" property, one can obtain scaling limits.

Other possible results by applying this theorem:

- Other simple boundary objects as bipartite maps with prescribed face degree sequence.
- Simple maps (without loops or multiple edges ) by extracting the 2-connected core.
- Maps with more than one simple hole, by treating one by one each hole.
- The precedents in higher genus.