A new proof of Aldous-Broder theorem

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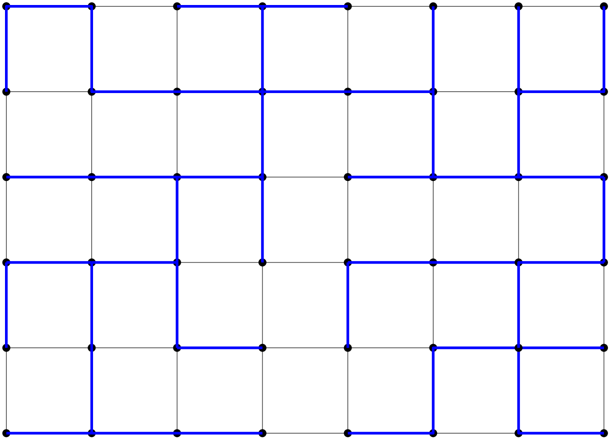
Definition (Spanning tree)

Given a graph G, we say that T is a spanning tree of G if it is a subgraph of G that is a tree containing all the vertices of G.



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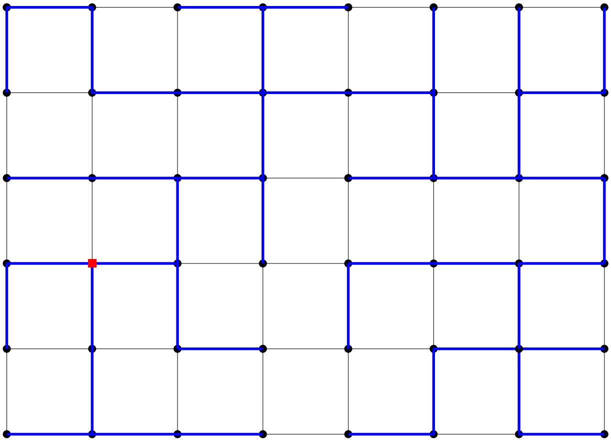

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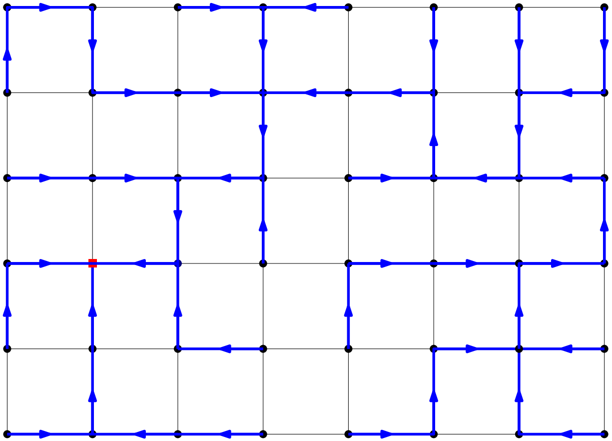

In this talk

I) Combinatorics on (weighted) spanning trees.
II) Given a graph G, how to sample a uniform spanning tree (UST)?
III) The original proof of Aldous-Broder theorem.
IV) New combinatorial proof of Aldous-Broder theorem.

I. Combinatorics: Counting (weighted) spanning trees

$\mathsf{ST}(G)$= set of spanning trees of G.

Matrix-tree theorem [Kirchhoff]

$$|\mathsf{ST}(G)| = \det\left(\mathsf{Laplacian}_G^{(r)}\right),$$

where $\mathsf{Laplacian}_G^{(r)}$ is the Laplacian matrix of G deprived of the line and column associated to r.

$$\mathsf{Laplacian}_G(i,j) = \left[deg(u_i)\mathbb{1}_{i=j} - |\{u_i,u_j\} \in E|\right]$$

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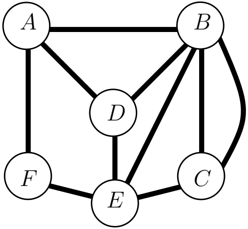
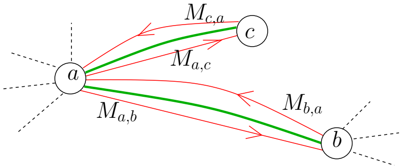


$$\mathrm{Laplacian_G} = \begin{pmatrix} 3 & -1 & 0 & -1 & 0 & -1 \\ -1 & 5 & -2 & -1 & -1 & 0 \\ 0 & -2 & 3 & 0 & -1 & 0 \\ -1 & -1 & 0 & 3 & -1 & 0 \\ 0 & -1 & -1 & -1 & 4 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$|\mathrm{ST}(G)| = \det\left(\mathrm{Laplacian}_G^{(A)}\right) = 98.$$

ST(G, r):= set of spanning trees of G rooted at r.
M := Markov kernel on G such that $\{u, v\} \in E \implies M_{u,v} > 0$ and $M_{v,u} > 0$.



$W(T, r) := \prod_{\vec{e} \in T} M_{\vec{e}}$ with edges pointing towards the root r.

Weighted Matrix-tree theorem [Kirchhoff]

$$\sum_{T \in \mathrm{ST}(G,r)} W(T, r) = \det\left((I - M)^{(r)}\right),$$

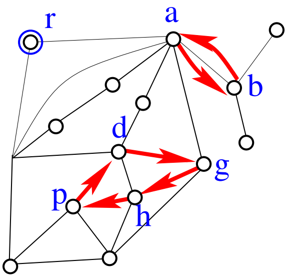
where $(I - M)^{(r)}$ is the matrix $(I - M)$ deprived of the line and column r.

Determinant expansion consequence

$$\det\left((I-M)^{(r)}\right) = \sum_{C\in\mathcal{C}} (-1)^{N(C)} \prod_{c \text{ cycles of } C} \prod_{\vec{e}\in c} M_{\vec{e}},$$

where the sum ranges over
$\mathcal{C}$ = set of collection of disjoint oriented cycles of length ≥ 1 avoiding r.

Define $\overleftarrow{M}_{x,y} := \rho_y M_{y,x}/\rho_x$, where ρ is the invariant measure associated to M.



$$\text{Weight} = (-1)^2(M_{a,b}M_{b,a})(M_{d,g}M_{g,h}M_{h,p}M_{p,d})$$

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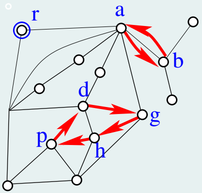
Define $\overleftarrow{M}_{x,y} := \rho_y M_{y,x}/\rho_x$, where ρ is the invariant measure associated to M.

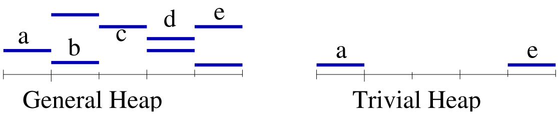
Claim

$$\det\left((I-M)^{(v)}\right) = \det\left((I-\overleftarrow{M})^{(v)}\right)$$



Heaps of pieces

Informally: some "elements" that are stacked.



General heap: (left) Equivalence class of words describing the history of the stack $= baeddecb = baeddceb = ebaddbce = \ldots.$

Trivial heap: (right) All the pieces on the ground $ae = ea$

Formally: a set of letters $\mathcal{P}$ is given and a binary relation R:

– $x \not R y$ means that x commutes with y (that is $xy = yx$),

– xRy means that x does not commute with y.

Heap of dominos: $\mathcal{P} = \{a, b, c, d, e\}$ aRb, bRc, cRd, dRe.

Heaps: Equivalence classes of words

$w \sim w'$ if they are equal up to a finite number of allowed commutations of consecutive letters.

Heap of pieces

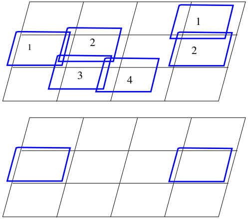


Figure: Heaps of squares. They do not commute if they share a side.

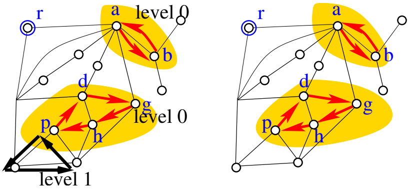


Figure: Heaps of cycles. They do not conmute if they share a vertex.

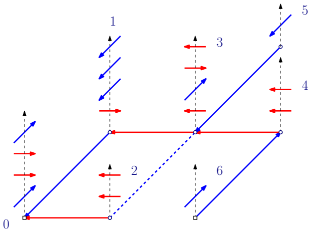


Figure: Heaps of outgoing edges. They do not commute if they start at the same point.

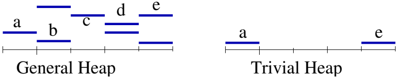


Figure: Heaps of dominoes. They do not commute if they share one extremity.

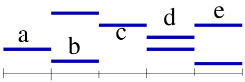
For a heap H

$$Weight(H) = \prod_{e \in H} w(e)$$

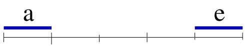
where $w : \mathcal{P} \to \mathbb{R}$ (or any formal commutative set)

Inversion lemma

$$\sum_{H \in \text{Heaps}} Weight(H) = \frac{1}{\sum_{H \in \text{TrivialHeaps}} (-1)^{|H|} Weight(H)}$$



General Heap



Trivial Heap

Example : Weight x for each piece, $Weight(H) = x^{|H|}$,

$$\sum_{H \in \text{Heaps}} Weight(H) = \frac{1}{1 - 5x + 6x^2 - x^3}$$

In particular for the heaps of cycles with weights given by M one has that

$$\sum_{HC \in \text{Heaps of cycles avoiding } r} W(HC) = \frac{1}{\sum_{HC \in \text{Trivial heaps of cycles avoiding } r} (-1)^{|HC|} W(HC)} = \det\left((I - M)^{(r)}\right)^{-1}$$

Summary of determinant formulas

1. **Matrix tree theorem (MTT):**

$$\sum_{T \in \mathrm{ST}(G,r)} \prod_{\vec{e} \in T} M_{\vec{e}} = \det\left((I - M)^{(r)}\right),$$

2. **Cycles expansion:**

$$\det\left((I - M)^{(r)}\right) = \sum_{HC \in \text{Trivial heaps of cycles avoiding } r} (-1)^{|H|} \prod_{\vec{e} \in H} M_{\vec{e}},$$

3. **Claim:**

$$\det\left((I - M)^{(r)}\right) = \det\left((I - \overleftarrow{M})^{(r)}\right)$$

Consequences

1. **Heaps of cycles:**

$$\sum_{HC \in \text{Heaps of cycles avoiding } r} W(HC) \overset{\text{(Inv. Lem.)}}{=} \det\left((I-M)^{(r)}\right)^{-1}$$

2. **Markov chain tree theorem:** the invariant measure of M satisfies

$$\rho_v \overset{\text{(Alg)}}{=} \frac{\det((I-M)^{(v)})}{Z} \overset{\text{(MTT)}}{=} \frac{\sum_{T \in \text{ST}(G,v)} \prod_{\vec{e} \in T} M_{\vec{e}}}{Z}$$

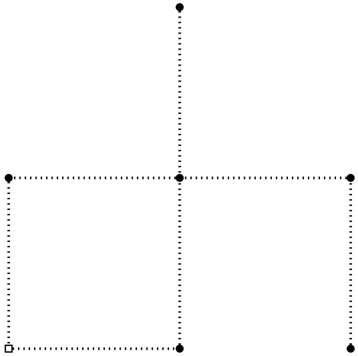
Important

$$Z \times \rho_v \times \left(\sum_{HC \in \text{Heaps of cycles avoinding } v} W(HC) \right) = 1$$

Consider a given graph G.
Algorithms to sample a UST:

1. Aldous-Broder algorithm.
2. Wilson algorithm.
3. Tutte polynomial + Matrix tree theorem.
4. ...

Consider an M-walk W in the invariant regime started at $r \in V$ up to the cover time.
Denote by **FirstEntrance**$(W) = (T, r)$, where r is the starting point of W and T is the spanning tree formed by the first edge used to visit each vertex.



Reversible: $\rho_u M_{u,v} = \rho_v M_{v,u}$

Theorem (Aldous-Broder ('89))

*For M positive and **reversible** Markov kernel with invariant distribution ρ. For any $T \in ST(G)$ one has*

$$\mathbb{P}\left(\textbf{\textit{FirstEntrance}}(W) = (T, r)\right) = \frac{\prod_{\vec{e} \in T} M_{\vec{e}}}{\sum_{w \in V} \det(I - M^{(w)})},$$

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Reversible Kernel: $\overleftarrow{M}_{x,y} = \rho_y / \rho_x M_{y,x}$

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From the claim we have that

$$\det\left(I - M^{(v)}\right) = \det\left(I - \overleftarrow{M}^{(v)}\right) \qquad \forall v \in V.$$

In particular, both normalization constants are the same.

Numerators are different when ρ is not reversible with respect to M.

The edges are directed from each node u toward its direct ancestor $a(u)$. For a tree $T \in \mathrm{ST}(G)$ and $r \in V$

$$\prod_{\vec{e} \in T} M_{\vec{e}} = \prod_{u \in T \setminus \{r\}} M_{u,a(u)} = \textbf{Const}.\ \rho_r \prod_{u \in T \setminus \{r\}} \rho_u M_{u,a(u)}$$

$$\prod_{\vec{e} \in T} \overleftarrow{M}_{\vec{e}} = \prod_{u \in T \setminus \{r\}} \left[M_{a(u),u}\, \rho_{a(u)}/\rho_u\right] = \textbf{Const}.\ \rho_r \prod_{u \in T \setminus \{r\}} \rho_{a(u)} M_{a(u),u}.$$

III. The Aldous-Broder proof: purely probabilistic!

$W = M$-walk in the invariant regime up to the cover time.

Denote by $\mathbf{LastExit}(W) = (T, r)$, where r is the ending point of W and T is the spanning tree formed by the last edge used to exit each vertex.

Fact

For w a deterministic walk up to the time bigger than the cover time one has

$$\mathbf{FirstEntrance}(w) = \mathbf{LastExit}(\overleftarrow{w})$$

III. The Aldous-Broder proof: purely probabilistic!

Consider the following rooted tree valued Markov chain $X_i = (T, r)$. To define X_{i+1} do as follows

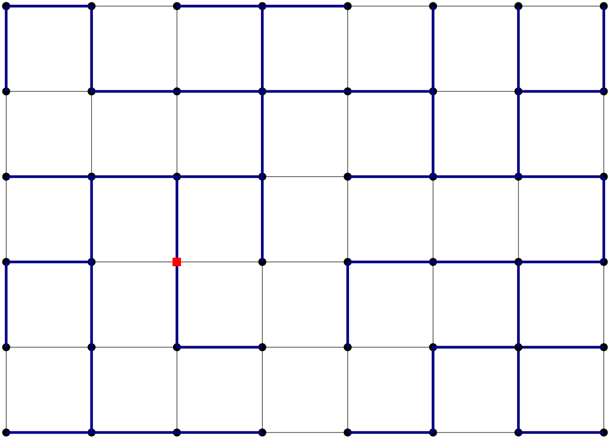

Figure: $X_i = (T, r)$

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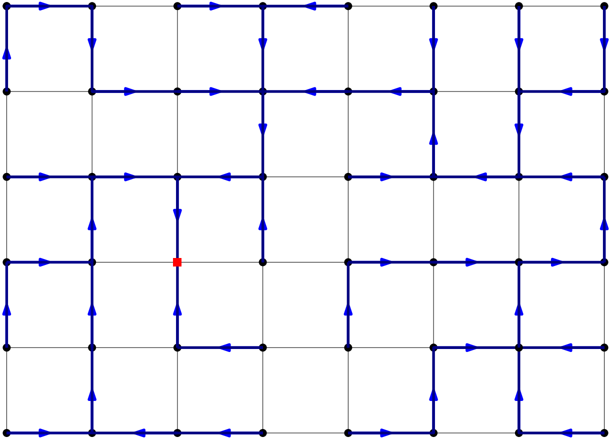

Figure: Orient the edges towards r

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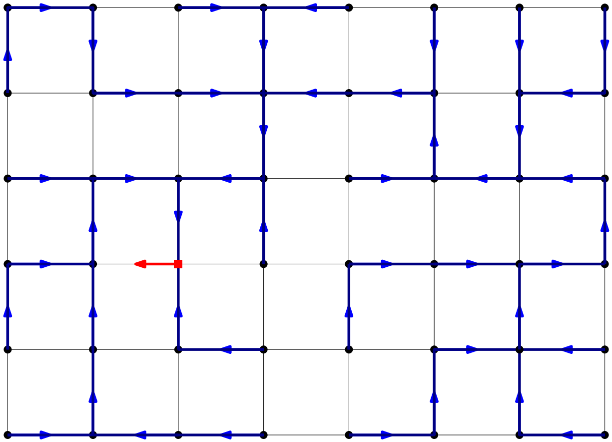

Figure: Make a step from the root following the kernel M.

Consider the following rooted tree valued Markov chain $X_i = (T, r)$. To define X_{i+1} do as follows

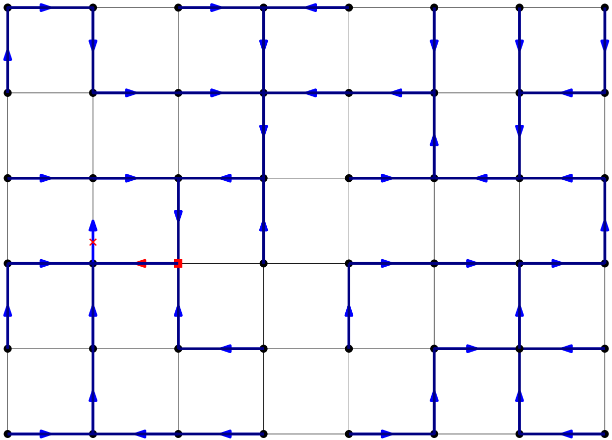

Figure: Suppress the outgoing edge in the destination point

Consider the following rooted tree valued Markov chain $X_i = (T, r)$. To define X_{i+1} do as follows

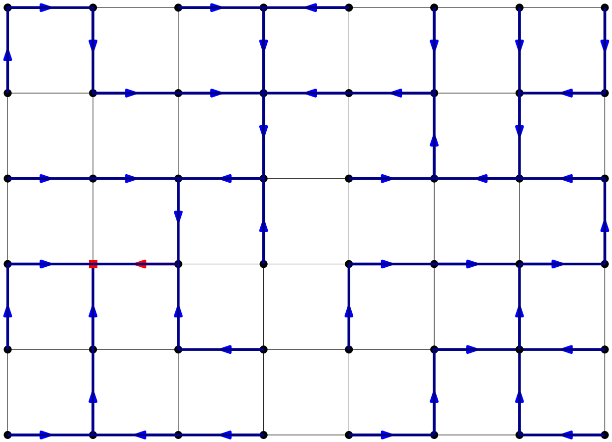

Figure: Change the root to the destination point

Consider the following rooted tree valued Markov chain $X_i = (T, r)$. To define X_{i+1} do as follows

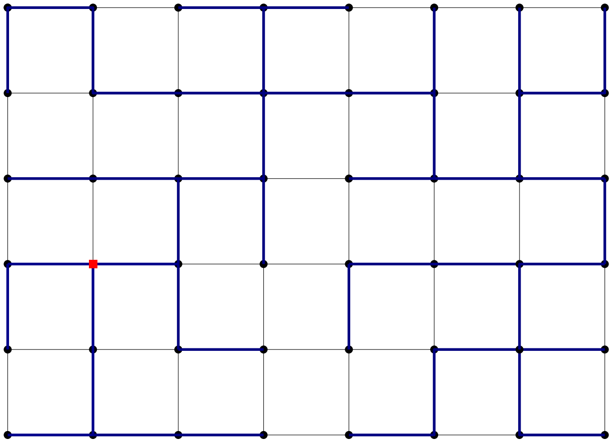

Figure: Define this resulting rooted tree as X_{i+1}

This rooted tree valued Markov chain has invariant measure

$$\nu(T,r) = \frac{\prod_{\vec{e}\in T} \overleftarrow{M}_{\vec{e}}}{\sum_{w\in V} \det(I - \overleftarrow{M}^{(w)})}$$

Aldous-Broder original proof: A coupling from the past idea

Run a $\overleftarrow{M}$-walk from $-\infty$, then use the tree valued Markov chain (**LastExit**) and reversibility to identify the distribution of the tree (**FirstEntrance**) at time 0.

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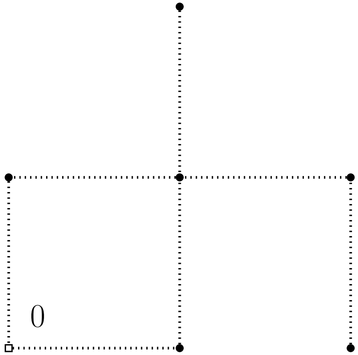
This proof is elegant and intricate. Nevertheless, no intuition of what is happening behind is left to work with.

Labelled extension



Denote by $\mathbf{Pionner}(W) = (\mathbf{FirstEntrance}(W), L)$ where L is the labeling.
$H_D(a, b) =$ probability starting from a that a walk following M escapes D at b.
$\overleftarrow{H}_D(a, b) =$ probability starting from a that a walk following $\overleftarrow{M}$ escapes D at b.

$$\begin{aligned}
&\mathbb{P}(\mathbf{Pionner}(W) = ((t, r), \ell)) \\
&= \mathbb{1}_{\ell_0 = r}\rho_{\ell_0} \prod_{i=0}^{n-2} \left[H_{\{\ell_{\leq i}\}}(\ell_i, a(\ell_{i+1}))M_{a(\ell_{i+1}),\ell_{i+1}}\right] \\
&= \left(\mathbb{1}_{\ell_0 = r}\rho_{\ell_{n-1}} \prod_{i=0}^{n-2} \left[\overleftarrow{H}_{\{\ell_{\leq i}\}}(a(\ell_{i+1}), \ell_i)\right] Z\right) \frac{\prod_{\vec{e}\in t} \overleftarrow{M}_{\vec{e}}}{Z}
\end{aligned}$$

Can we prove using combinatorics that

$$\sum_{\ell} \mathbb{1}_{\ell_0 = r} Z \rho_{\ell_{n-1}} \prod_{i=0}^{n-2} \left[\overleftarrow{H}_{\{\ell_{\leq i}\}}(a(\ell_{i+1}), \ell_i) \right] = 1?$$

(the sum ranges over all decreasing labellings of the tree)

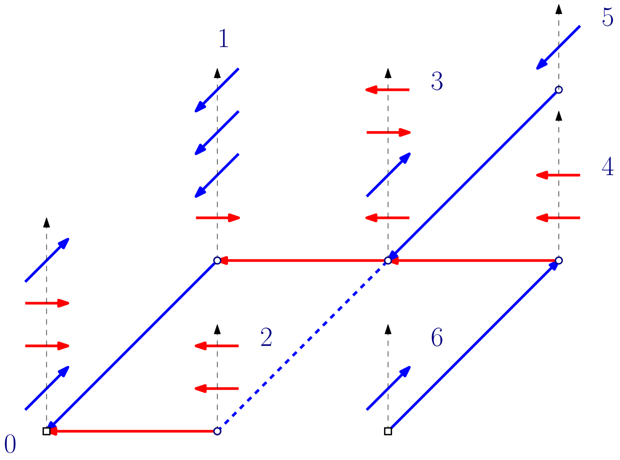


Figure: Path seen backward as a heap of outgoing edges

IV. New proof

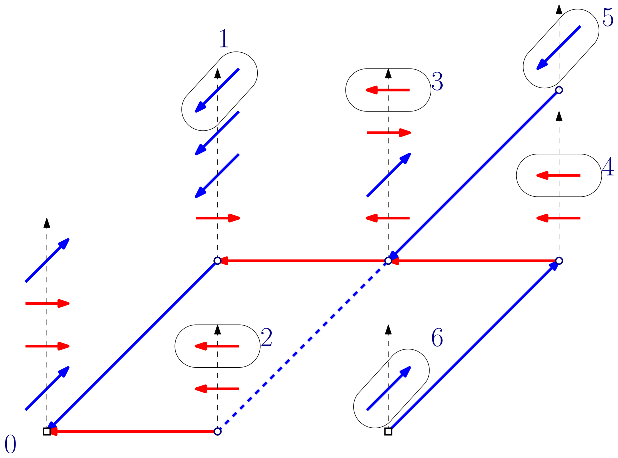


Figure: The tree edges are always on top of the piles.

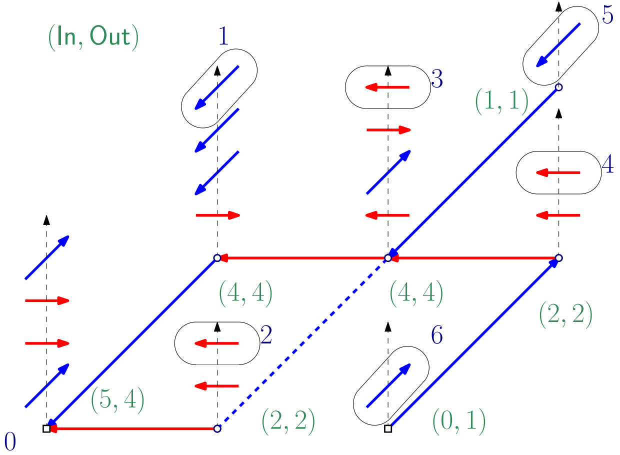


Figure: Count the incoming and outgoing edges

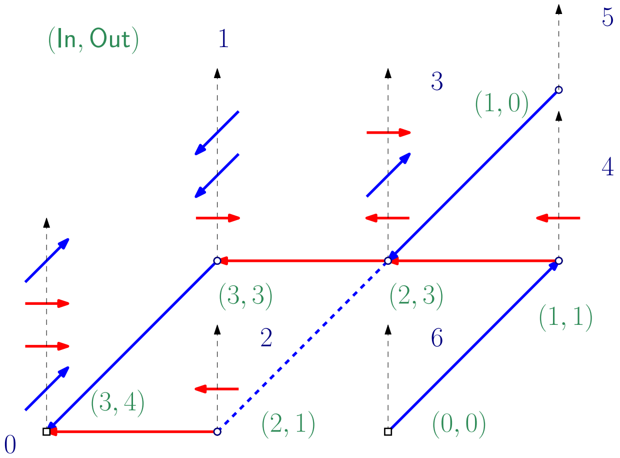


Figure: Pop-out the tree edges to construct H^{-t} (update (In,Out))

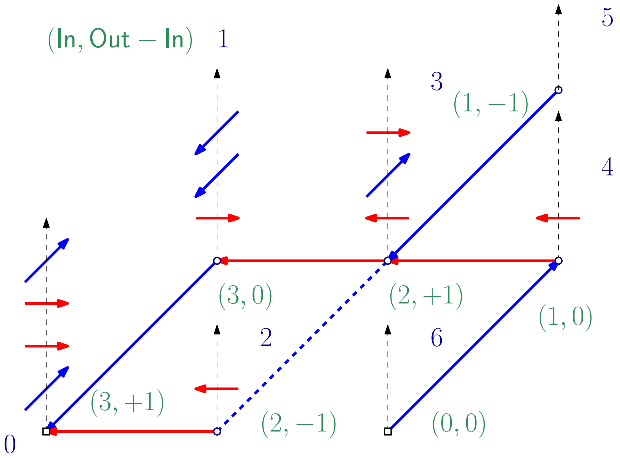


Figure: Convenient to keep an eye on (In,Out-In)

IV. New proof

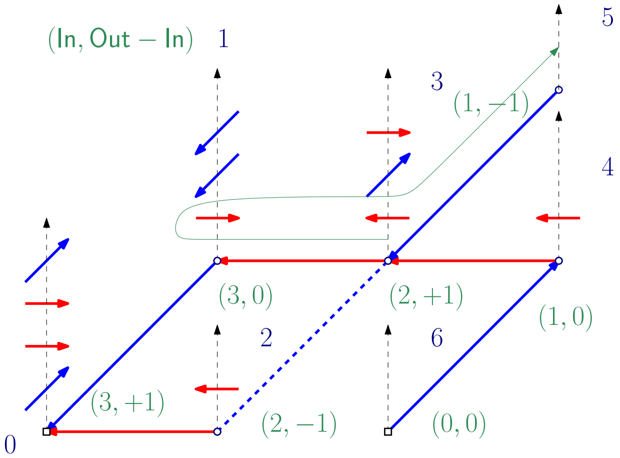


Figure: Play golf!

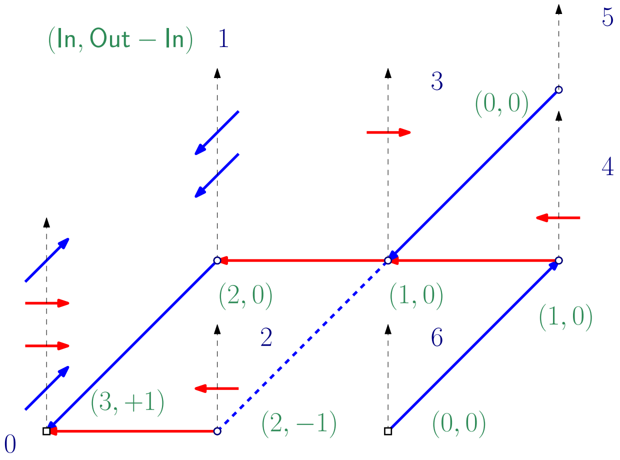


Figure: Supress the path and update (In,Out-In)

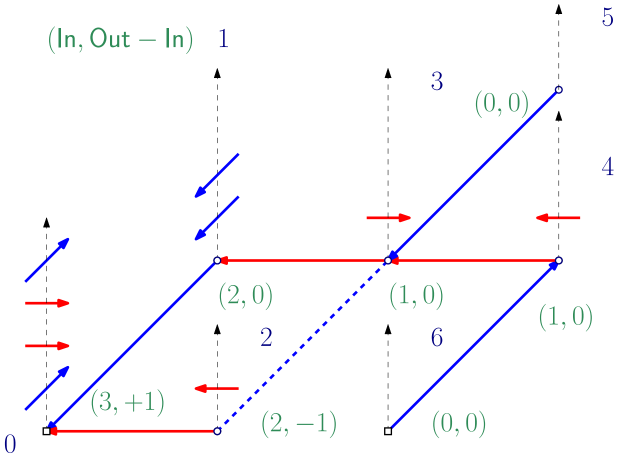


Figure: Let the pieces fall

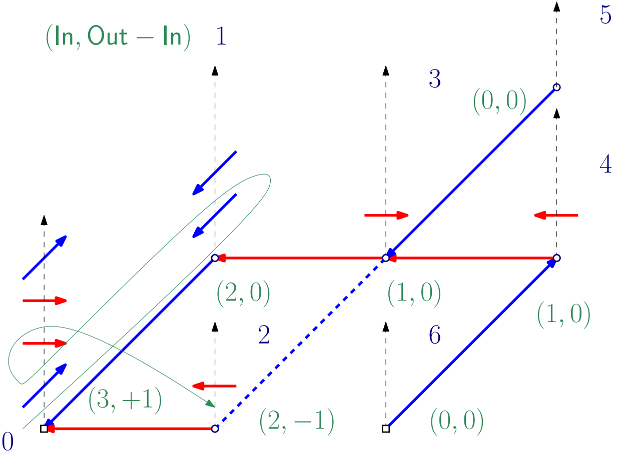
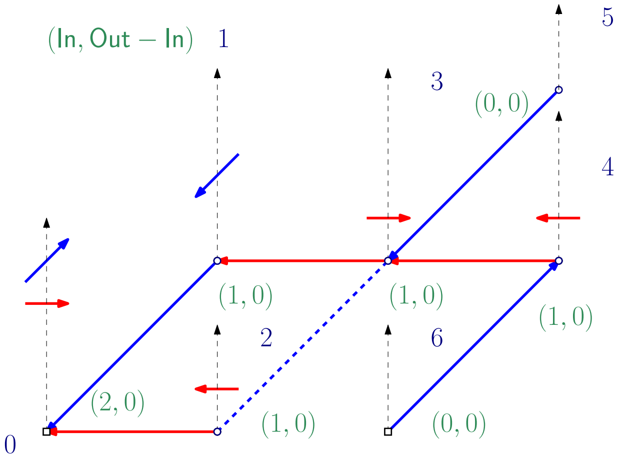


Figure: Continue playing golf with next emitting vertex.



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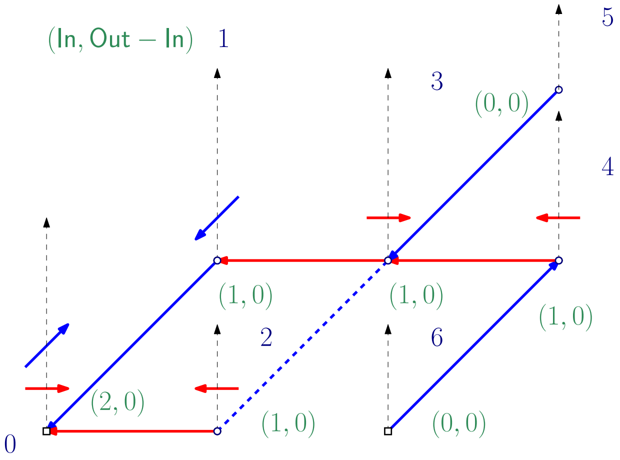

Figure: Let the pieces fall

IV. New proof

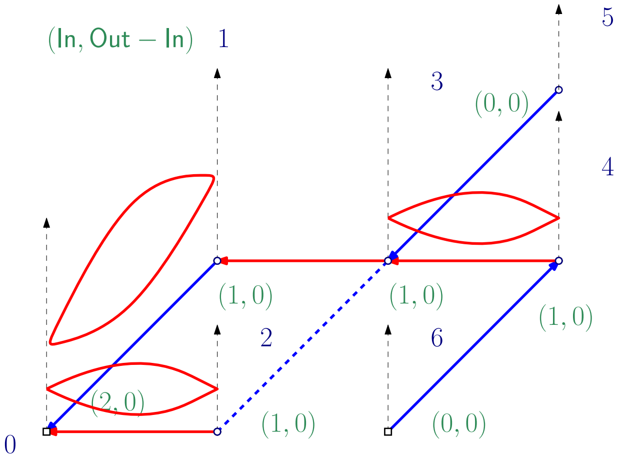

Figure: heap of cycles

IV. New proof

The heap of outgoing edges H^{-t} is a heap only on $V \setminus \ell_{n-1}$ and $H^{-t} = \text{Golf} \times HC$. Recall we fix a treatment order to fix an ordering of the starting points in the Golf game.

$$\sum_{\ell} \mathbb{1}_{\ell_{\mathbf{0}}=r} Z \rho_{\ell_{n-\mathbf{1}}} \times \prod_{i=0}^{n-2} \left[\overleftarrow{H}_{\ell_{\leq i}}(a(\ell_{i+1}), \ell_i) \right]$$

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$$= \sum_{H^{-t} \text{ valid}} Z\rho_{\ell_{n-\mathbf{1}}} \times W(H^{-t})$$

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$$\begin{aligned}
&\sum_{\ell} \mathbb{1}_{\ell_0 = r} Z \rho_{\ell_{n-1}} \times \prod_{i=0}^{n-2} \left[\overleftarrow{H}_{\ell_{\leq i}}(a(\ell_{i+1}), \ell_i) \right] \\
&= \sum_{H^{-t} \text{ valid}} Z \rho_{\ell_{n-1}} \times W(H^{-t}) \\
&= \sum_{(Golf, HC) \text{ valid}} Z \rho_{\ell_{n-1}} \times W(Golf) \times W(HC)
\end{aligned}$$

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&= \sum_{(Golf, HC) \text{ valid}} Z\rho_{\ell_{n-\mathbf{1}}} \times W(Golf) \times W(HC) \\
&= \underbrace{\sum_{Golf \text{ valid}} W(Golf)}_{=1} \times \underbrace{Z\rho_{\ell_{n-\mathbf{1}}} \times \left(\sum_{\substack{HC \in \textbf{ heaps of cycles} \\ \text{avoiding } \ell_{n-\mathbf{1}}}} W(HC) \right)}_{\overset{\textbf{Important}}{=} 1}
\end{aligned}
$$

The first by a probabilistic algorithm.

Corollary (F.-Marckert ('21+))

If W is a SRW stopped when $m < |V|$ vertices has been discovered, then the tree ***FirstEntrance***(W) *is not uniform in the set of subtrees of G of size m.*

Consider τ_A as the hitting time of the set A. Define for a rooted tree (T, r), the ancestor of v towards the root as $a(v)$.

Proposition (F.-Marckert ('21+))

For any spanning tree T of G we have

$$\sum_{\ell} \prod_{i=0}^{n-2} \mathbb{P}_{a(\ell_{i+1})}\left(\overleftarrow{\tau}_{\{\ell_i\}} < \overleftarrow{\tau}_{\{\ell_{i+1},\ldots,\ell_{n-1}\}}\right) = 1,$$

where the sum ranges over the set of decreasing labelling of (T, r).
Moreover, this is not true if T is not a spanning tree.

Motivation: Odded Schramm question

Conformally invariant scaling limits: an overview and a collection of problems

Oded Schramm

2.5. Lattice trees. We now present an example of a discrete model where we suspect that perhaps conformal invariance might hold. However, we do not presently have a candidate for the scaling limit.

Fix $n \in \mathbb{N}_+$, and consider the collection of all trees contained in the grid G that contain the origin and have n vertices. Select a tree T from this measure, uniformly at random.

Problem 2.8. What is the growth rate of the expected diameter of such a tree? If we rescale the tree so that the expected (or median) diameter is 1, is there a limit for the law of the tree as $n \to \infty$? What are its geometric and topological properties? Can the limit be determined?

It would be good to be able to produce some pictures. However, we presently do not know how to sample from this measure.

Problem 2.9. Produce an efficient algorithm which samples lattice trees approximately uniformly, or prove that such an algorithm does not exist.

Figure: Schramm ICM 2006.

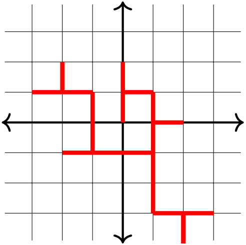
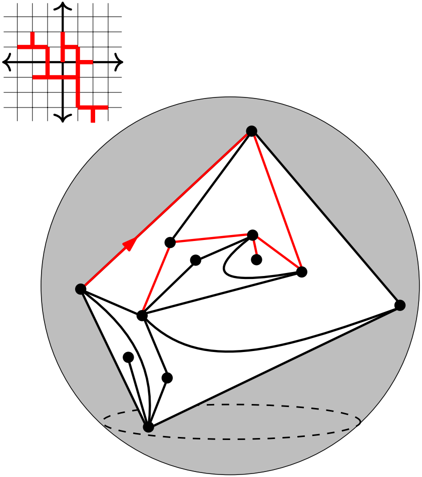
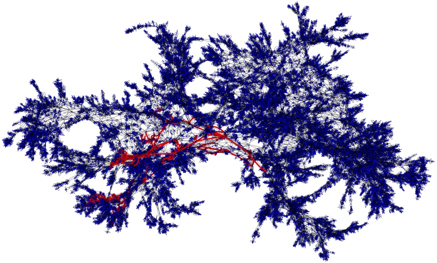


Figure: Subtree of size 20 containing the origin on $\mathbb{Z}^2$.



(a) tree-decorated quad. 10 faces, tree of size 6.

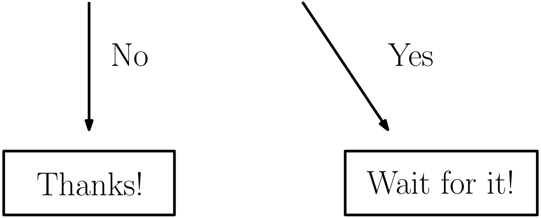


(b) Unif. tree-decorated quad. 90k faces and tree of size 500.

We try to contribute to Schramm's question in different ways:

- Trying to generalize known algorithms to a target size.
- Sampling (approx.) from the uniform measure in the set of subtrees of given size.
- Estimate scaling exponents.
- A new combinatorial proof of the Aldous-Broder theorem.

Do I have more time?



V. Wilson (LERW version)

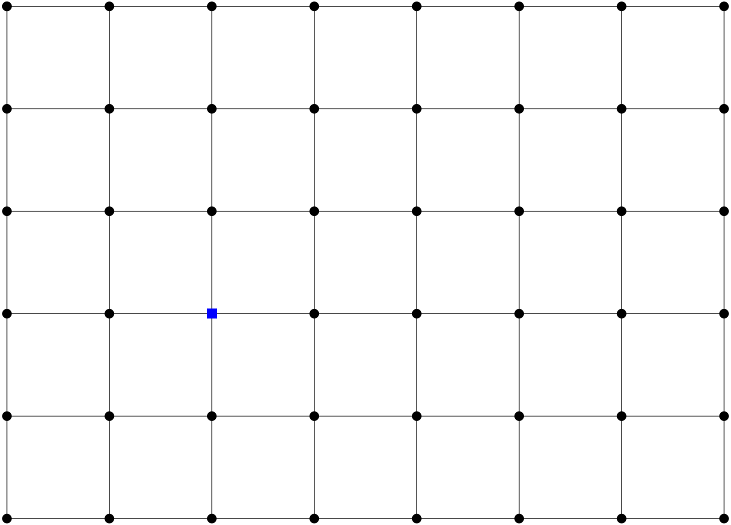


Figure: Pick any vertex r as root (square vertex)

V. Wilson (LERW version)

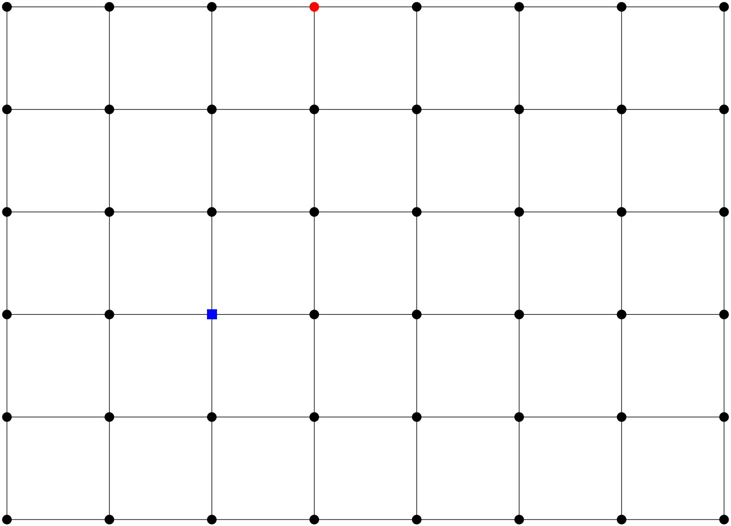


Figure: Pick another vertex $v \in V \setminus \{r\}$.

V. Wilson (LERW version)

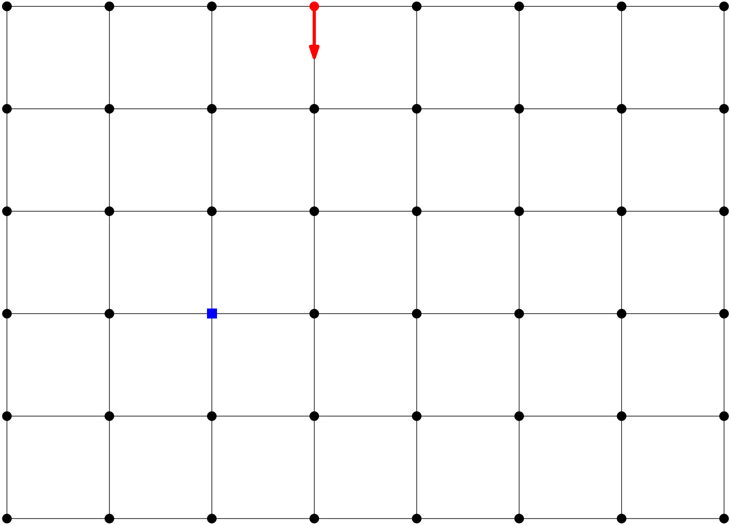
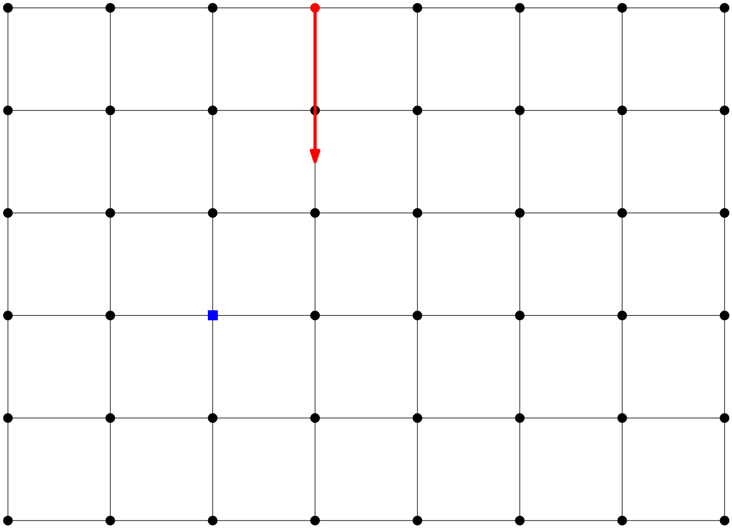
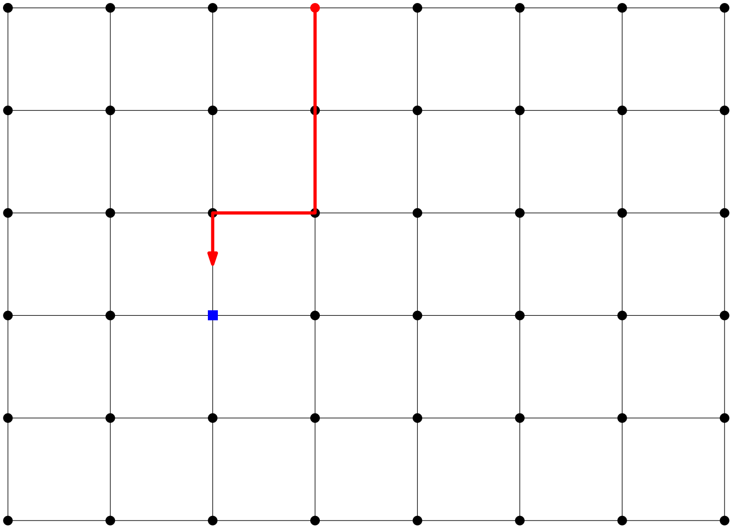


Figure: Start a loop erased random walk (LERW) from v until it hits r.

V. Wilson (LERW version)



V. Wilson (LERW version)



V. Wilson (LERW version)

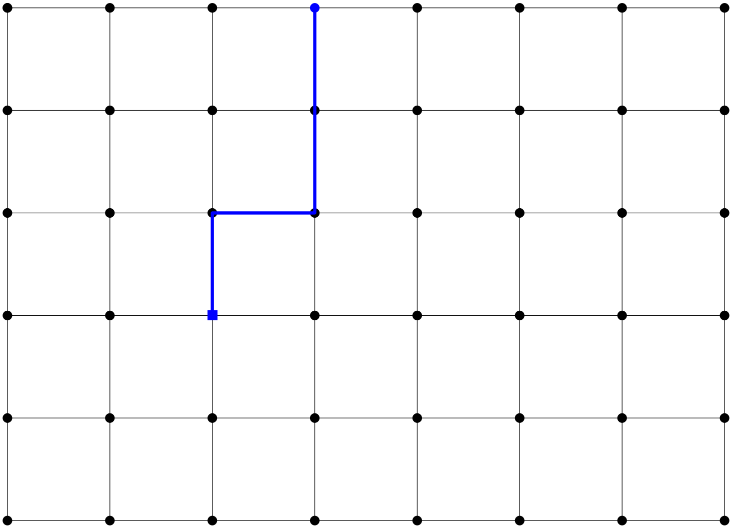


Figure: Set this path as the current tree T

V. Wilson (LERW version)

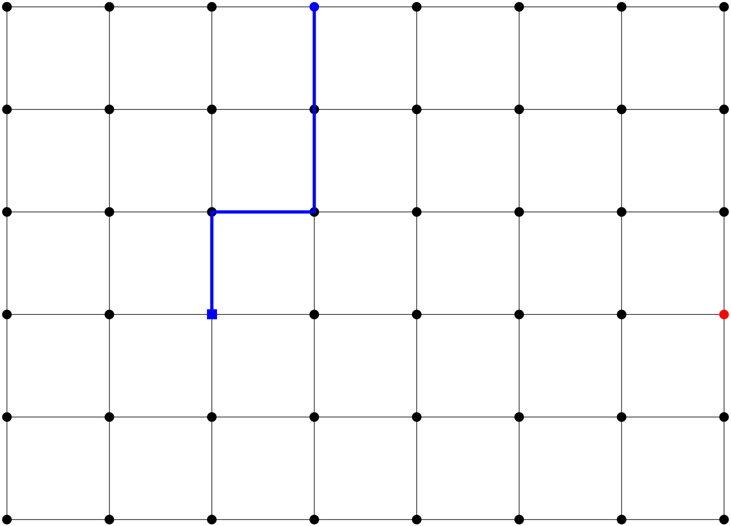


Figure: Pick another vertex $v \in V \setminus V(T)$.

V. Wilson (LERW version)

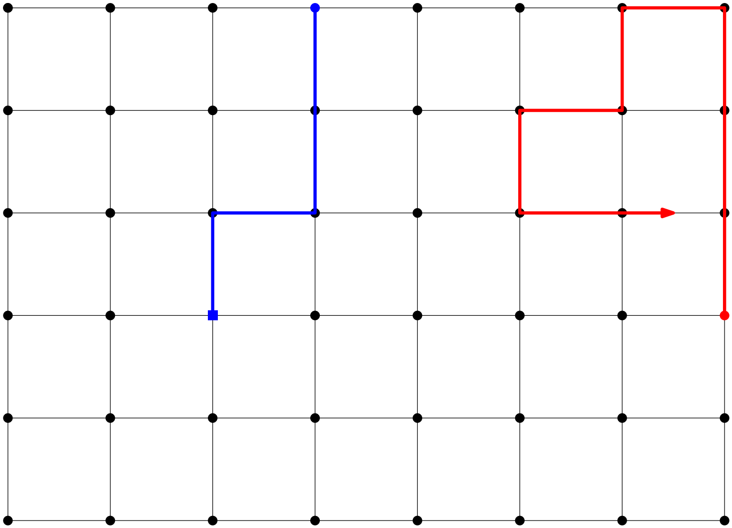


Figure: Start a LERW from v until it hits T.

V. Wilson (LERW version)

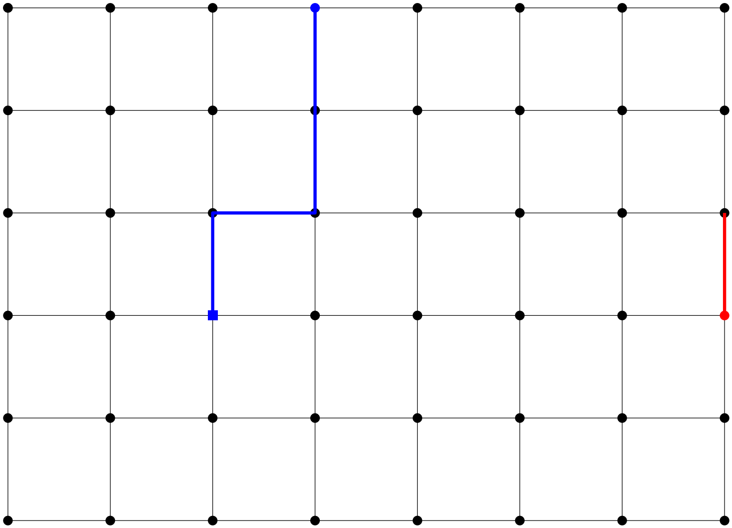
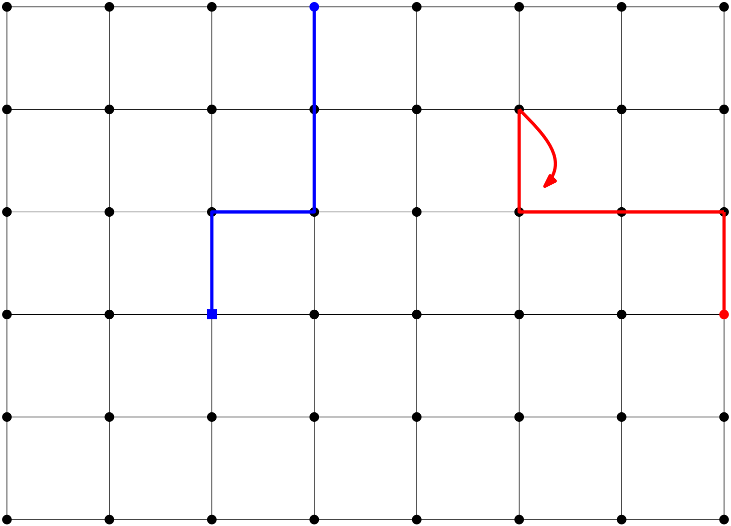
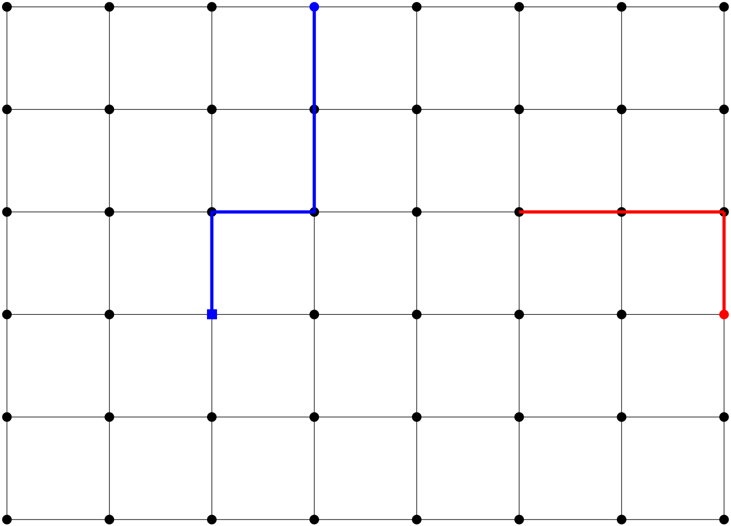


Figure: LERW: A cycle is created → throw it away and continue.

V. Wilson (LERW version)



V. Wilson (LERW version)



V. Wilson (LERW version)

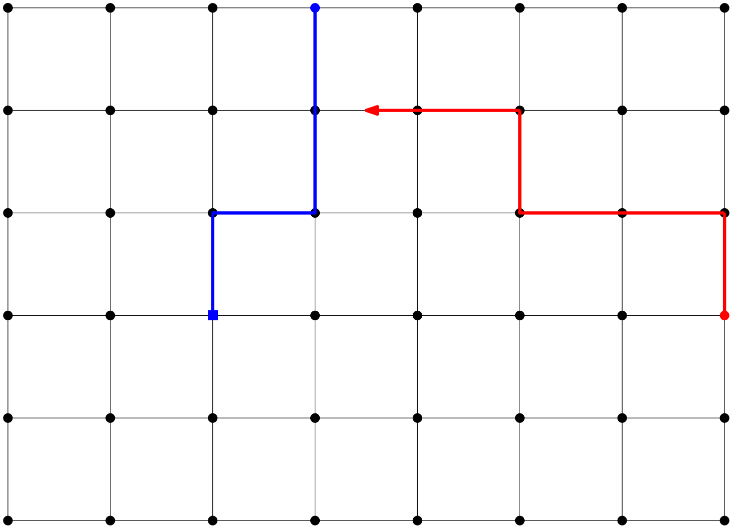
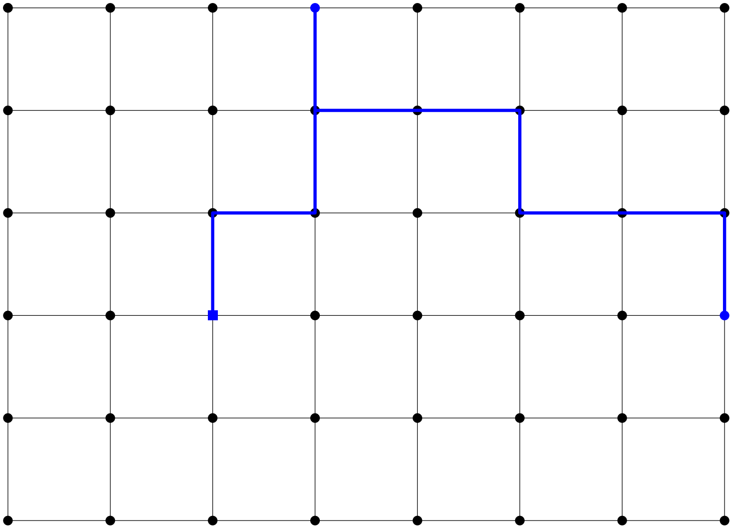
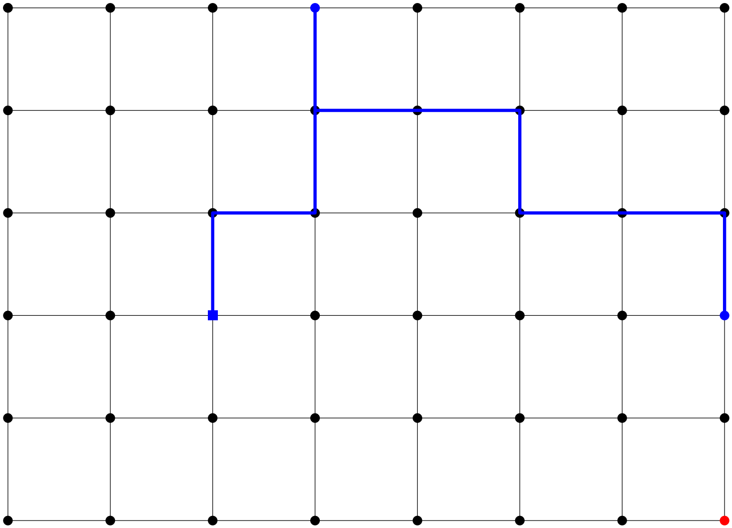


Figure: you got it!

V. Wilson (LERW version)



V. Wilson (LERW version)



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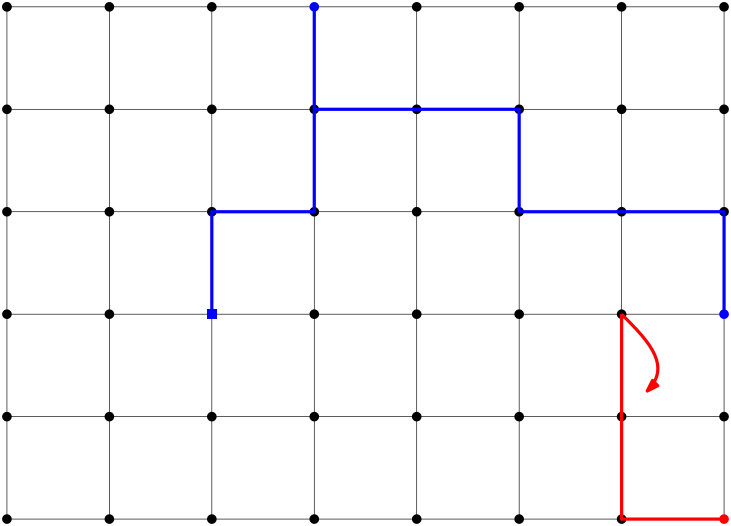

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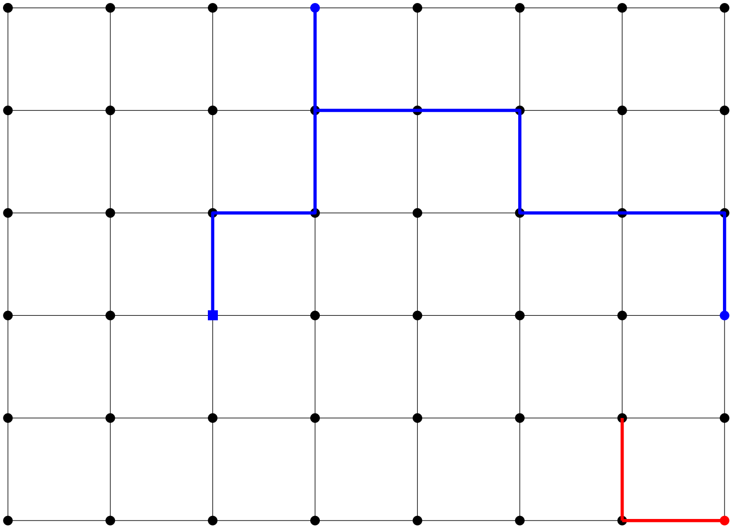

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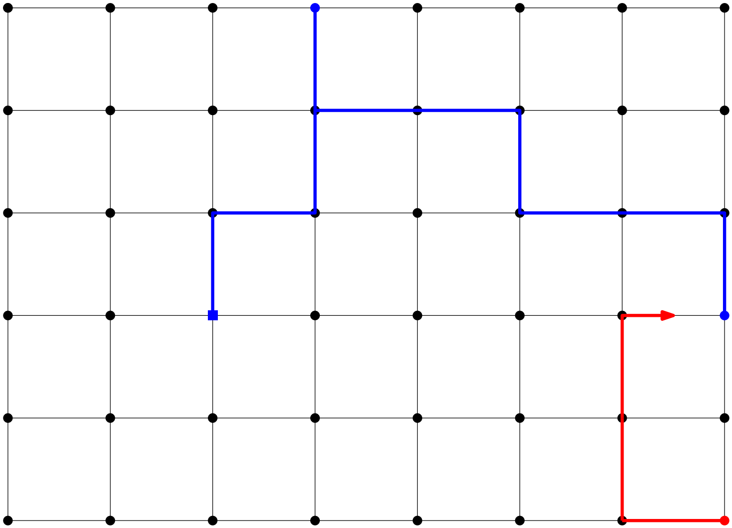

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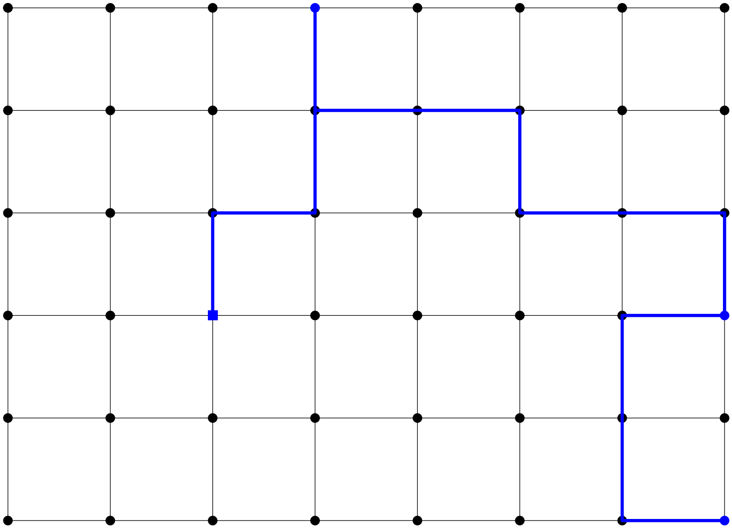

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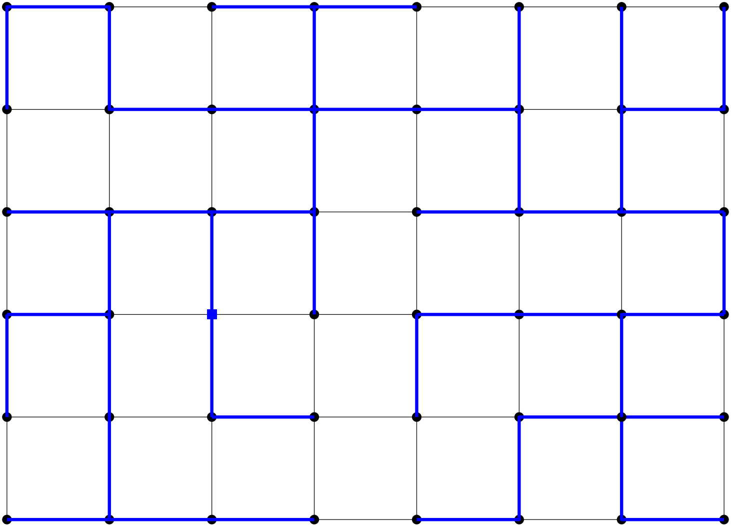


Figure: Final tree T.

V. Wilson (LERW version)

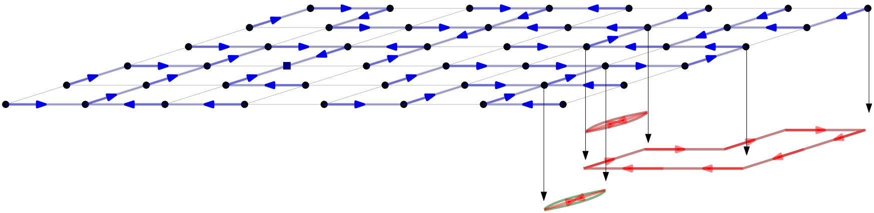


Figure: Heaps of cycles and a tree : $HC \times T$.

Call $(\mathcal{HC}, \mathcal{T})$ the r.v. associated to the heap of cycles and rooted tree of the previous algorithm.

Theorem (Wilson ('96))

For any finite graph the algorithm ends with probability 1. Moreover, for any heap of cycles HC and any tree $T \in ST(G, r)$ one has

$$\mathbb{P}\left((\mathcal{HC}, \mathcal{T}) = (HC, T)\right) = W(HC) \times W(T),$$

Important: Expected running time.

- Wilson: mean hitting time.
- Aldous-Broder: expected cover time (always greater than or equal to the mean hitting time).

Both Wilson and Aldous-Broder algorithms use random walks to construct trees.

Natural question: Can Wilson be coupled with the same random walk used by Aldous-Broder?

From enumeration to uniform sampling

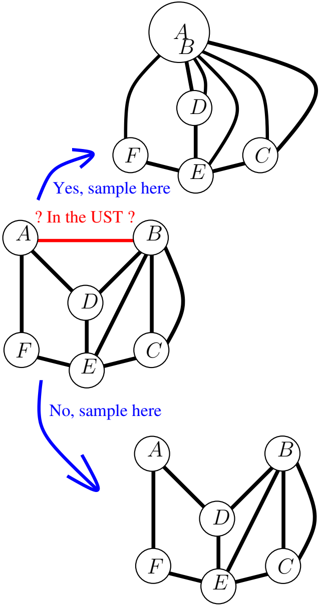
– fix an edge e of G.

$$\mathbb{P}(e \in \mathsf{UST}(G)) = \frac{|\mathsf{ST}(G.e)|}{|\mathsf{ST}(G)|} = \frac{\text{Determ...}}{\text{Determ...}}$$

$\rightarrow$ add e to the spanning tree with this probability (and contract the edge e in G),

$\rightarrow$ otherwise, delete e from G.

Drawback: Not fast, $|\mathsf{ST}(G)|$ is huge, and the program has to deal with huge numbers.



THANKS!