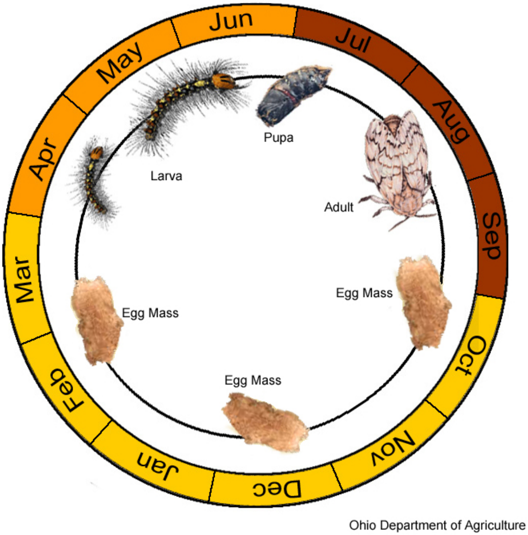
Survival and coexistence for spatial population models with forest fire epidemics.

Luis Fredes
(With A. Linker (U. Chile) and D. Remenik (U. Chile).)

Random trees and graphs, Luminy, 2019

Figure: Gypsy moth.

Figure: Egg masses.



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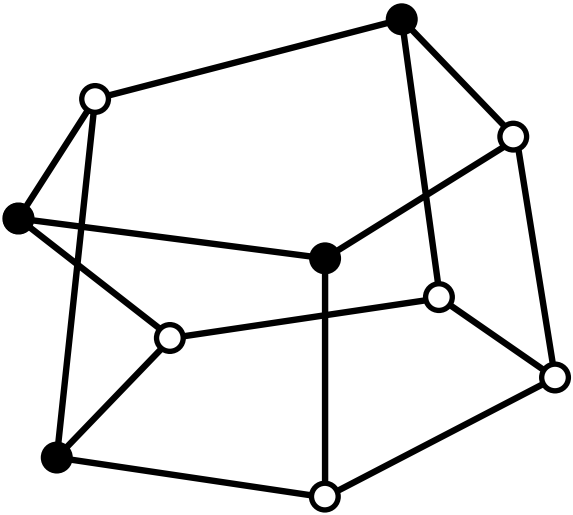


Figure: Configuration at time t. **Moth living period**.

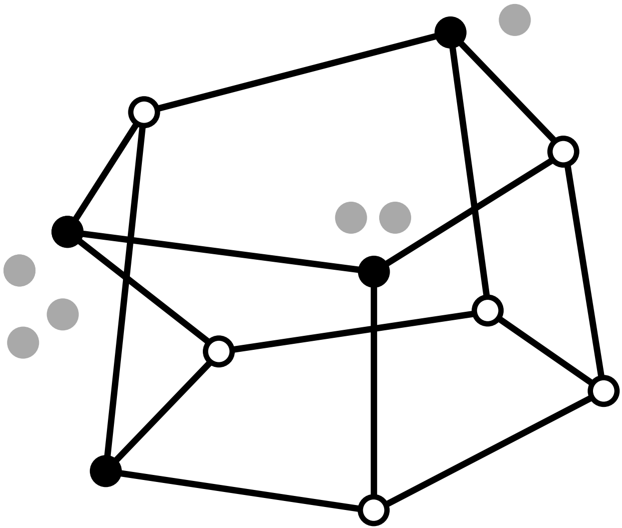


Figure: Growth stage configuration time t. **Random offspring of mean** β.

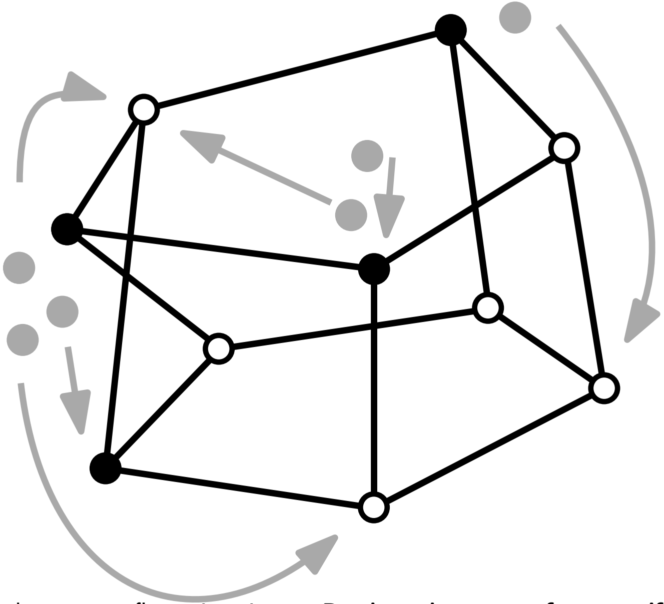


Figure: Growth stage configuration time t. **Random placement of eggs, uniformly in V_N for each egg**.

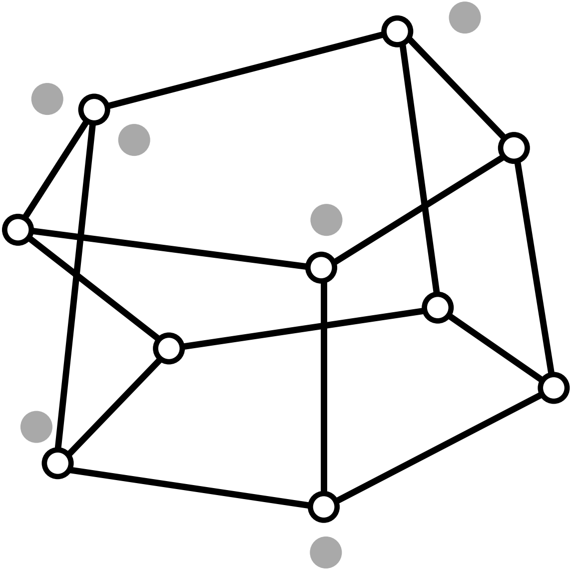


Figure: Growth stage configuration time t. **Moth die and assignation of sites is done**.

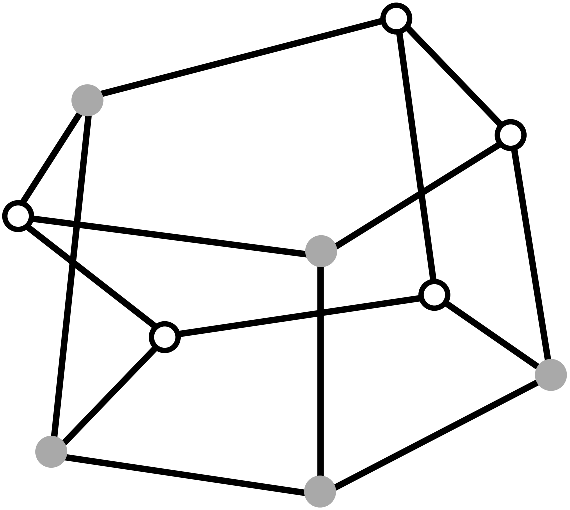


Figure: Growth stage configuration time t. **If there is more than one, only one survives (not enough room)**.

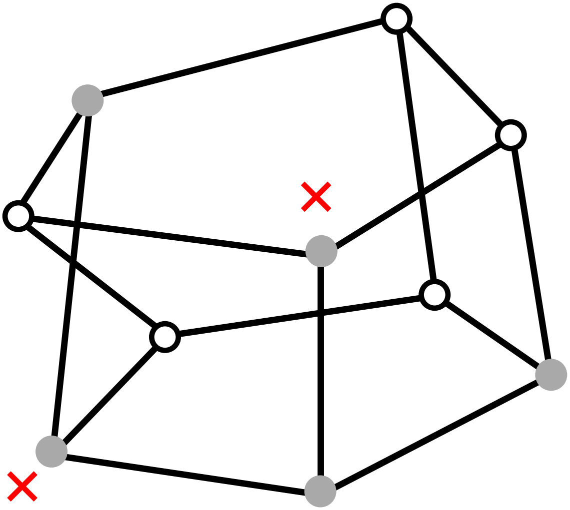


Figure: Epidemic stage configuration time $t + 1/2$. **Epidemic attacks with probability α_N each site, independently**.

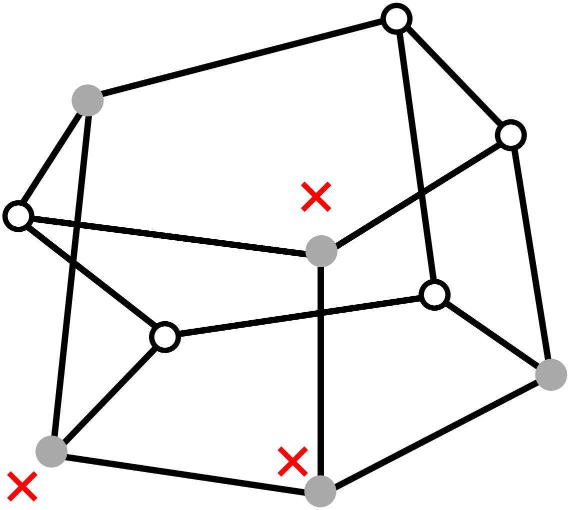
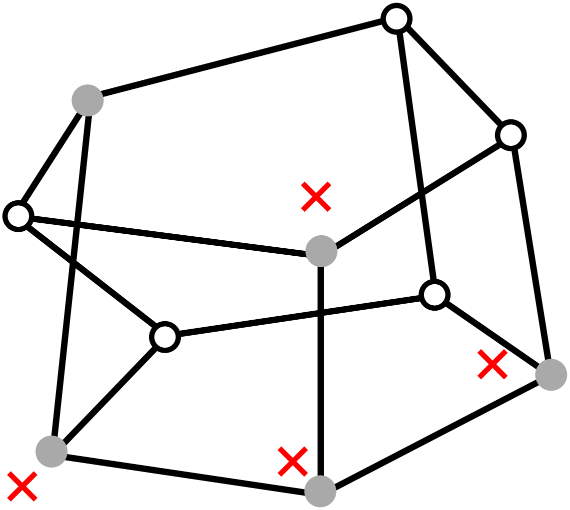


Figure: Epidemic stage configuration time $t + 1/2$. **Spreading of epidemic**.



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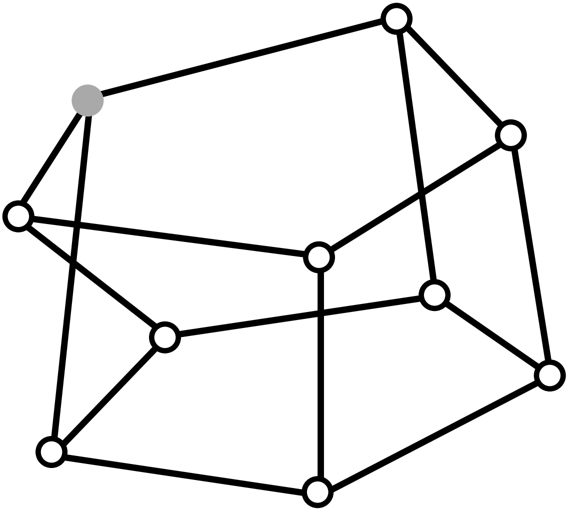


Figure: Epidemic stage configuration time $t + 1/2$. **Survivors**.

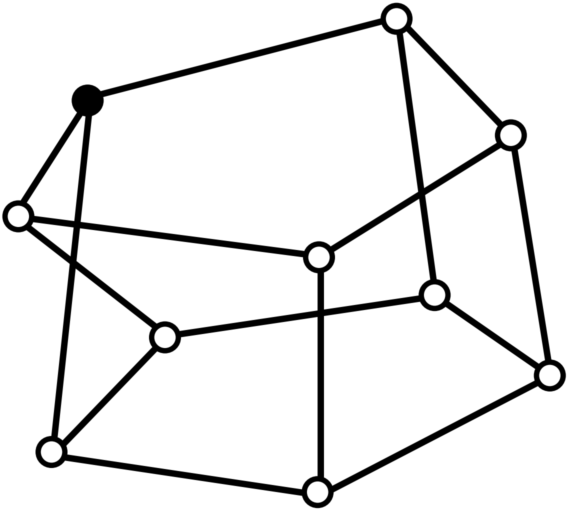
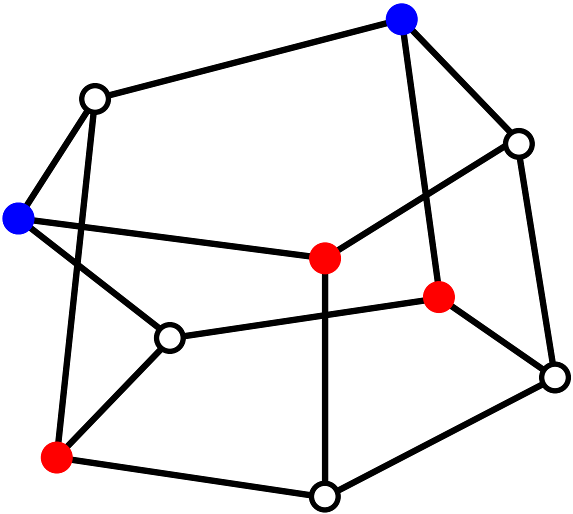


Figure: Configuration time $t + 1$. **Moth living period**.

Multi-type moth model



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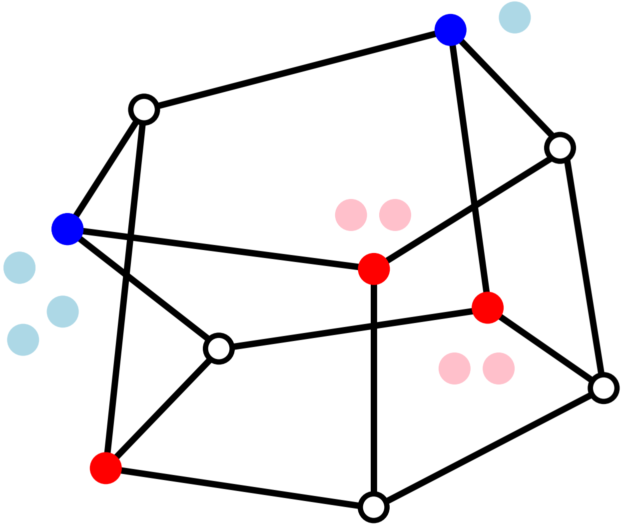

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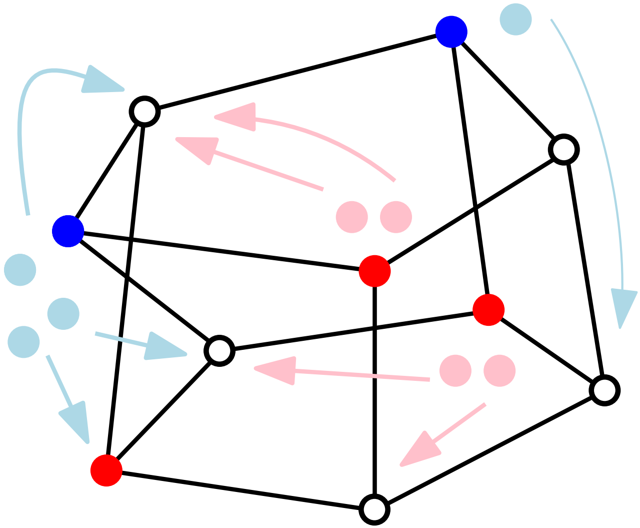

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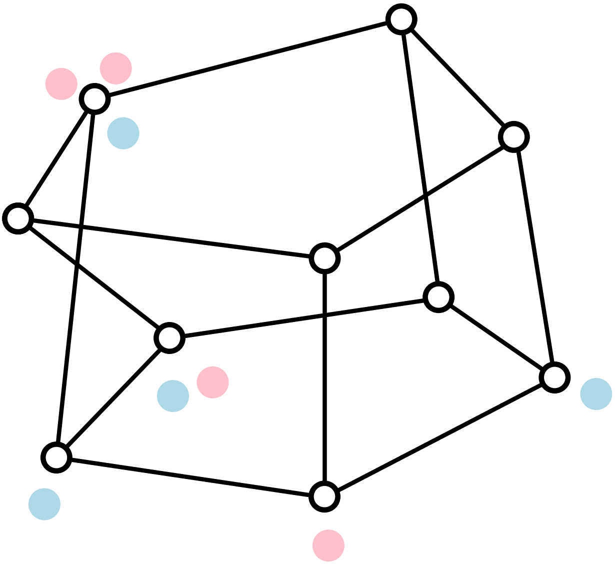

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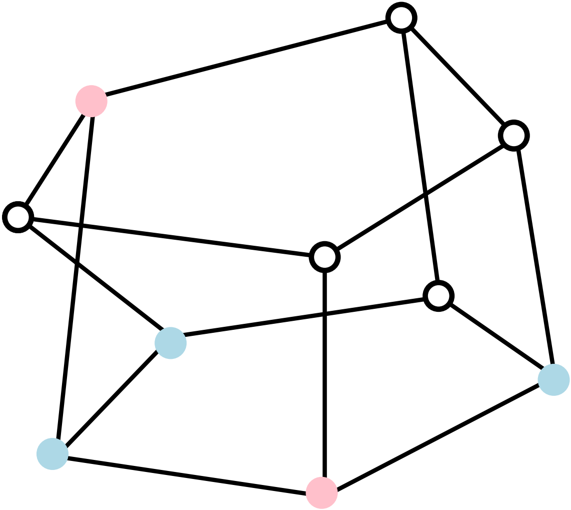


Figure: Epidemic stage configuration time $t + 1/2$. **The type is assigned uniformly among all eggs that arrived to each vertex**.

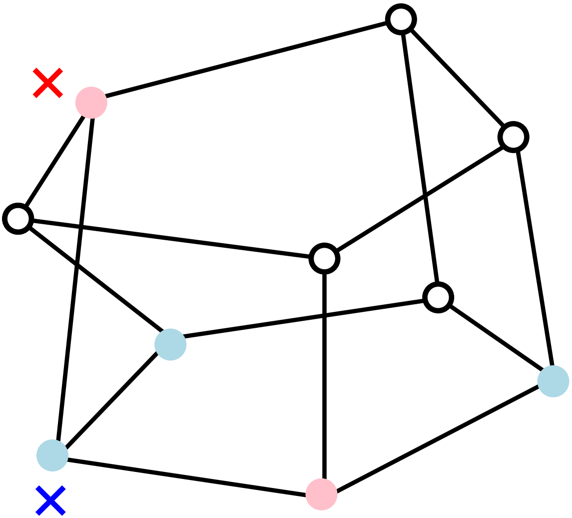
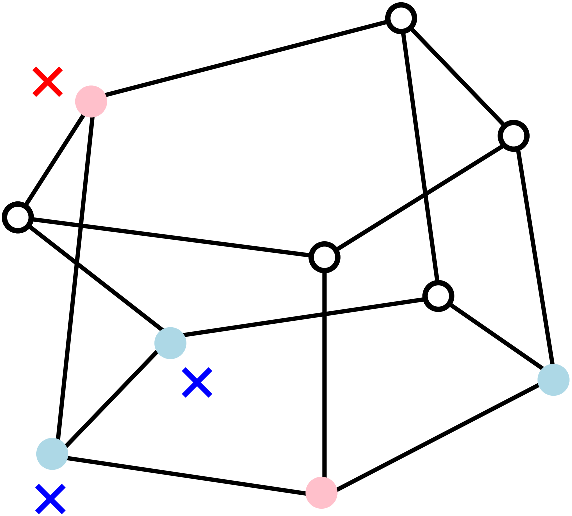


Figure: Epidemic stage configuration time $t + 1/2$. **Epidemics attack with probability $\alpha_N(i)$ each site of type i, independently.**



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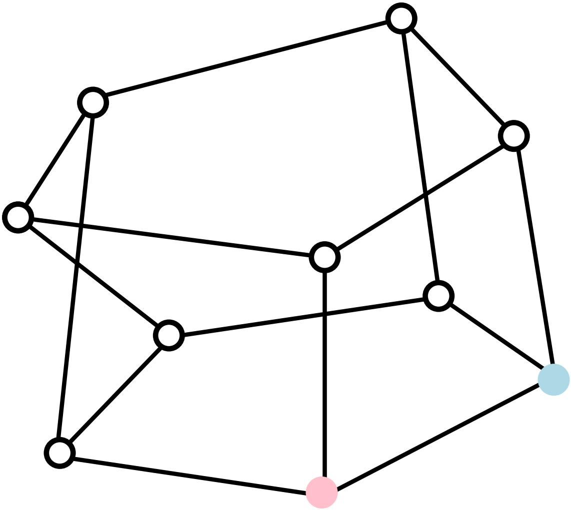

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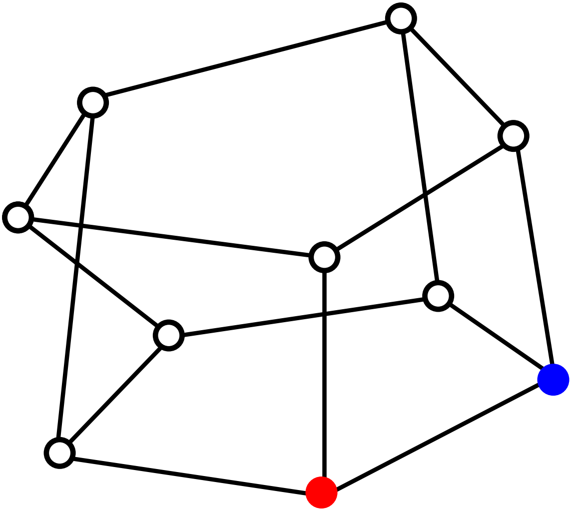

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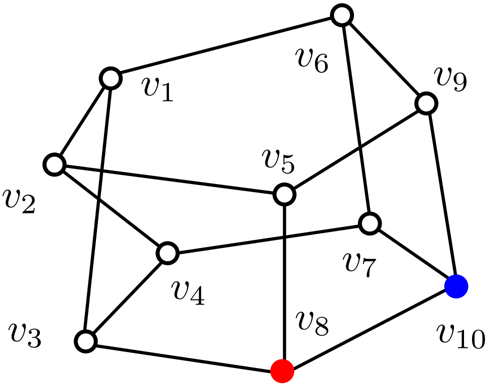

$$\eta_{t+1} = (\eta_{t+1}(v_1), \ldots, \eta_{t+1}(v_{10}))$$
$$= (0, 0, 0, 0, 0, 0, 0, 1, 0, 2)$$

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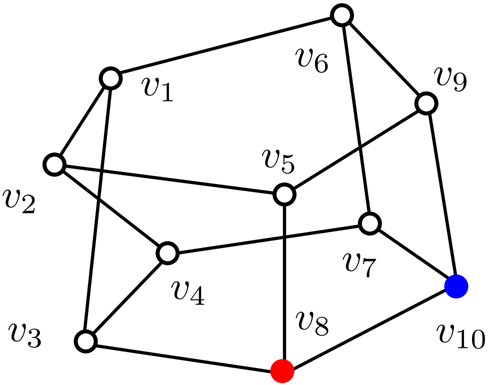

$$\rho_{t+1} = (\rho_{t+1}(1), \rho_{t+1}(2))$$
$$= (1,1)/10$$

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Spoiler alert!

Forest fires epidemics **change** this behavior.

Multi-type moth model

Consider a graph $G_N = (V_N, E_N)$ with N vertices and $m \in \mathbb{N}^*$.
The MMM is a discrete time Markov process $\left(\eta_k\right)_{k\geq 0}$ is defined using an initial configuration $\eta_0 \in \{0, 1, \ldots, m\}^{V_N}$ and 2 families of parameters:

1. $$\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{R}_+^m$$

2. $$\vec{\alpha}_N = (\alpha_N(1), \alpha_N(2), \ldots, \alpha_N(m)) \in [0, 1]^m.$$

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Growth:

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Density vectors by $((\rho_k^N(1), \rho_k^N(2), \ldots, \rho_k^N(m)), k \geq 0)$ defined by

$$\rho_k^N(i) := \frac{1}{N} \sum_{x \in V_N} \mathbb{1}_{\{\eta_k^N(x) = i\}}, \quad i \in \{1, 2, \ldots, m\}$$

Previous results $m = 1$

When $m = 1$ the parameters are no longer vector, so we write α_N and β.

Theorem (Durrett & Remenik '09)

Suppose $m = 1$ and $(G_N)_{N \in \mathbb{N}}$ a sequence of random uniform 3-regular graphs. Assume that the infection probability satisfies

$$\alpha_N \to 0 \quad \textit{and} \quad \alpha_N \log(N) \to \infty, \quad \textit{as} \quad N \to \infty$$

and also

$$\rho_0^N \xrightarrow{(d)} p_0 \in [0, 1] \quad \textit{as} \quad N \to \infty.$$

Then the process $(\rho_k^N)_{k \geq 0}$ converges in distribution as $(N \to \infty)$ to the (deterministic) orbits $(p_k)_{k \geq 0}$ of an explicit dynamical system started at p_0.

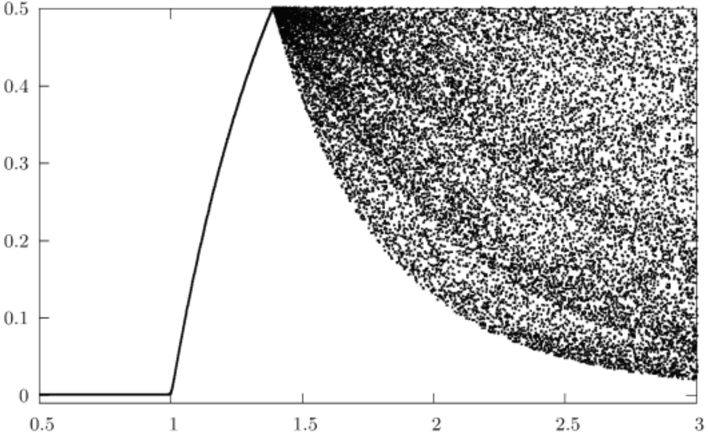

Figure: Bifurcation diagram in β for the dynamical system.

First extension : $m = 1$ and $\alpha_N \to \alpha \in (0,1)$

Proposition (F-Linker-Remenik '18)

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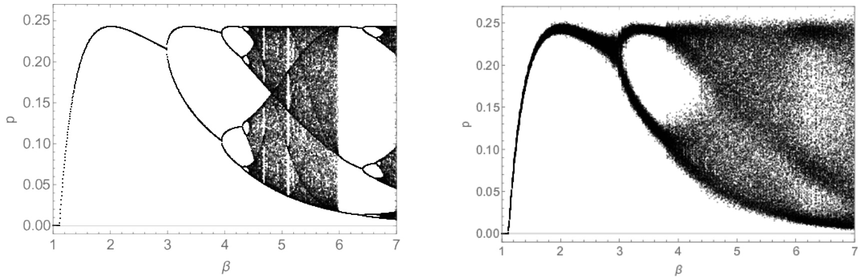

Figure: Left: Bifurcation diagram in β for the dynamical system with fixed $\alpha = 0.1$. Right: stochastic process simulations densities for $\alpha = 0.1$ and different β's.

Define the **effective offspring parameter**

$$\phi(\alpha_N, \beta) = \beta(1 - \alpha),$$

and **the extinction time**

$$\tau_N = \inf\{k \geq 0 : \rho_k^N = 0\}.$$

Theorem (F-Linker-Remenik '18)

- **Extinction:** *When $\phi(\alpha_N, \beta) < 1$ there is $C > 0$ independent of N such that*

$$\mathbb{E}(\tau_N) \leq C \log(N). \tag{1}$$

- ***Survival:*** *If $\phi(\alpha_N, \beta) > 1$ and ρ_0^N the initial density is bounded away from 0, then there exists $c > 0$ (depending only on ρ_0^N and α_N) such that*

$$\mathbb{E}(\tau_N) \geq cN. \tag{2}$$

Convergence $m \geq 2$

Vectors again!

Theorem (F-Linker-Remenik '18)

Consider $m \geq 2$ and $\vec{\alpha} \in [0,1]^m$ (epidemic parameters). If

$$\vec{\alpha}_N \to \vec{\alpha} \quad \text{and} \quad \alpha_N(i)\log(N) \to \infty \quad \text{as} \quad N \to \infty, \ \forall i \in \{1, ..., m\},$$

and also

$$\vec{\rho}_0^{\,N} \xrightarrow{(d)} \vec{p}_0 \in [0,1] \quad \text{as} \quad N \to \infty.$$

Then, the sequence of density vectors $(\vec{\rho}_k, k \geq 0)$ converges for the product topology to the orbits

$$(\vec{p}_k, k \geq 0)$$

of an explicit dynamical system depending on $\vec{\beta}$ and $\vec{\alpha}$.

Survival and coexistence dynamical system $m = 2$

Proposition (F-Linker-Remenik '18)

There are explicit regions of the parameter space giving either domination (black/white regions) or coexistence (gray region) for the dynamical system.

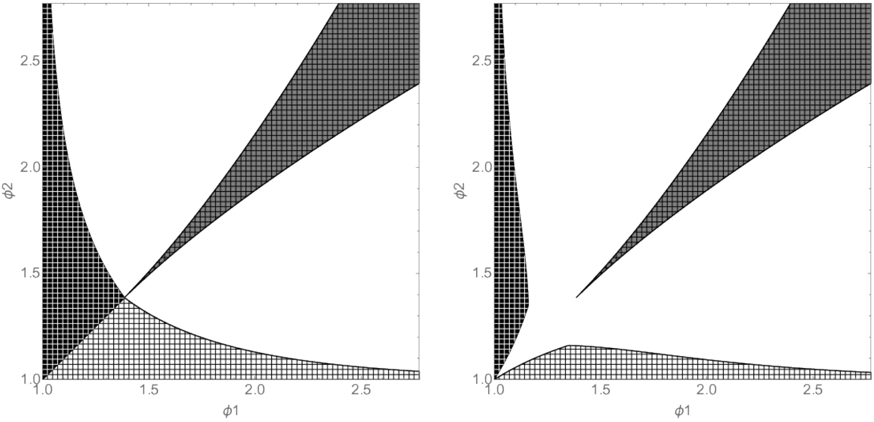

Figure: Left $\alpha(1) = \alpha(2) = 0$ and right $\alpha(1) = \alpha(2) = 0.1$.

Survival and coexistence of the stochastic process $m = 2$

Define $\bar{\alpha} := \min\{\alpha(1), \alpha(2)\}$.

Theorem (F-Linker-Remenik '18)

For $m = 2$, assume that $\vec{\rho}_0^N \to \vec{p}_0$. Then, under some technical condition:

1. *In the **domination regime** (of the dynamical system):*
   - *The weakest type dies out in time of order $\log(N)$.*
   - *The strongest one survives for at least order*

$$\begin{cases} e^{\sqrt{\log(N)}} & \text{if } \bar{\alpha} = 0 \\ N^{\bar{\alpha}/5} & \text{if } \bar{\alpha} > 0. \end{cases}$$

2. *In the **coexistence regime** (of the dynamical system):*
   - *Both types survive for at least order*

$$\begin{cases} e^{\sqrt{\log(N)}} & \text{if } \bar{\alpha} = 0 \\ N^{\bar{\alpha}/5} & \text{if } \bar{\alpha} > 0. \end{cases}$$

Extensions

In words:

1. We proved the results when each egg is placed uniformly in $\mathcal{N}_N(x) = B(x, r_N)$ with $r_N \to \infty$ at a certain rate.
2. We can work with d-regular graphs instead.
   Difficulties: the explicit dynamical system turns out to be ugly and the regimes of **domination** and **coexistence** cannot be treated at once for a generic d.
3. We think that for $m = 1$ the survival regime satisfies an expected absorption time of exponential order.

Thanks for your attention!