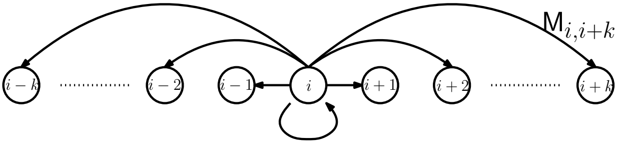
Almost triangular Markov chains on $\mathbb{N}$

Luis Fredes
(Work with J.F. Marckert)

IMB

Transition matrix: $M = [M_{i,j}]_{i,j\in S}$ with non-negative real entries that sum up to one on each row. A Markov chain Y with transition matrix M satisfies

$$\mathbb{P}(X_{n+1} = b | X_n = a) = M_{a,b}.$$



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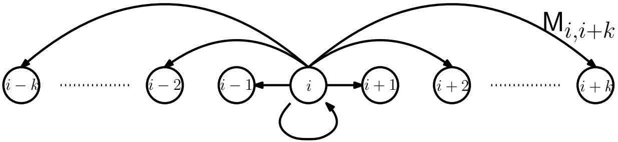

Irreducible: every pair $a, b \in S$ has a finite sequence $\ell = \ell(a,b)$ of steps with positive probability ($M^{\ell}_{a,b} > 0$) such that b can be reached from a.

From now on we assume irreducibility

Recurrence / Transience: A chain with transition matrix M is called recurrent if for all/one state $a \in S$ the probability to return to a is one, otherwise the chain is called transient.

Positive recurrence: The expected return time of all/one state $a \in S$ is finite, i.e. $\mathbb{E}_a(\tau_a^+) < +\infty$.

Invariant measure: A measure π on S is said to be invariant by M if,

$$\sum_{a \in S} \pi_a M_{a,b} = \pi_b \quad \text{for all } b \in S.$$

Motivation

Almost-upper triangular ◹ and almost-lower triangular ◺

$$U := \begin{bmatrix} U_{0,0} & U_{0,1} & U_{0,2} & U_{0,3} & U_{0,4} & \cdots \\ U_{1,0} & U_{1,1} & U_{1,2} & U_{1,3} & U_{1,4} & \cdots \\ 0 & U_{2,1} & U_{2,2} & U_{2,3} & U_{2,4} & \cdots \\ 0 & 0 & U_{3,2} & U_{3,3} & U_{3,4} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots \end{bmatrix}, \quad L := \begin{bmatrix} L_{0,0} & L_{0,1} & 0 & 0 & 0 & \cdots \\ L_{1,0} & L_{1,1} & L_{1,2} & 0 & 0 & \cdots \\ L_{2,0} & L_{2,1} & L_{2,2} & L_{2,3} & 0 & \cdots \\ L_{3,0} & L_{3,1} & L_{3,2} & L_{3,3} & L_{3,4} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots \end{bmatrix}$$

Birth and death processes (BDP)

$$T = \begin{bmatrix} T_{0,0} & T_{0,1} & 0 & 0 & 0 & \cdots \\ T_{1,0} & T_{1,1} & T_{1,2} & 0 & 0 & \cdots \\ 0 & T_{2,1} & T_{2,2} & T_{2,3} & 0 & \cdots \\ 0 & 0 & T_{3,2} & T_{3,3} & T_{3,4} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots \end{bmatrix}$$

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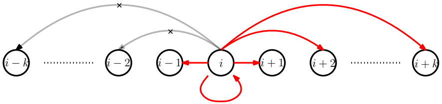

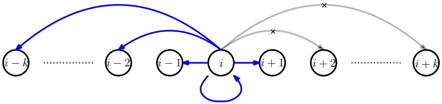

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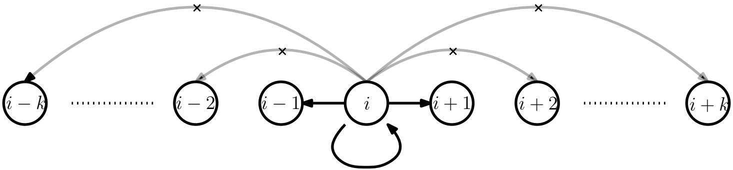

Theorem: Tridiagonal case (Karlin & McGregor '57)

The following measure π with $\pi_0 = 1$

$$\pi_a = \prod_{j=1}^{a} \frac{\mathsf{T}_{j-1,j}}{\mathsf{T}_{j,j-1}} \quad \text{for all } a \geq 1,$$

is the **unique** invariant by T up to a constant factor and the chain is

- **positive recurrent** if and only if

$$\sum_{k\geq 1} \prod_{j=1}^{k} \frac{\mathsf{T}_{j-1,j}}{\mathsf{T}_{j,j-1}} < +\infty$$

- **recurrent** if and only if

$$\sum_{k\geq 0} \prod_{j=1}^{k} \frac{\mathsf{T}_{j,j-1}}{\mathsf{T}_{j,j+1}} = +\infty.$$

Theorem: Almost-upper triangular case (F. -Marckert '21)

The following measure π with $\pi_0 = 1$

$$\pi_a := \frac{\det(\mathrm{Id} - \mathrm{U}_{[0,a-1]})}{\prod_{j=1}^{a} \mathrm{U}_{j,j-1}} \quad \text{for all } a \geq 1.$$

is the **unique** invariant by U up to a constant factor and the chain is

- **positive recurrent** if and only if

$$\sum_{a=1}^{\infty} \frac{\det\left(\mathrm{Id} - \mathrm{U}_{[0,a-1]}\right)}{\prod_{j=1}^{a} \mathrm{U}_{j,j-1}} < \infty.$$

- **recurrent** if and only if

$$\lim_{b \to +\infty} \mathrm{U}_{1,0} \frac{\det(\mathrm{Id} - \mathrm{U}_{[2,b-1]})}{\det(\mathrm{Id} - \mathrm{U}_{[1,b-1]})} = 1.$$

Plan of the talk

- Proof of the theorem for Almost-upper triangular.
- Almost-lower triangular.
- Link between almost-upper and almost-lower.
- Recovering the results of BDP with our results.
- Another model.

Combinatorial warm up I: Matrix tree theorem

ST(G)= set of spanning trees of G.

Matrix-tree theorem [Kirchhoff]

$$|\mathrm{ST}(G)| = \det\left(\mathrm{Laplacian}_G^{(r)}\right),$$

where $\mathrm{Laplacian}_G^{(r)}$ is the Laplacian matrix of G deprived of the line and column associated to r.

$$\mathrm{Laplacian}_G(i,j) = \left[deg(u_i)\mathbb{1}_{i=j} - |\{u_i, u_j\} \in E|\right]$$

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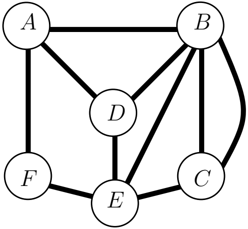
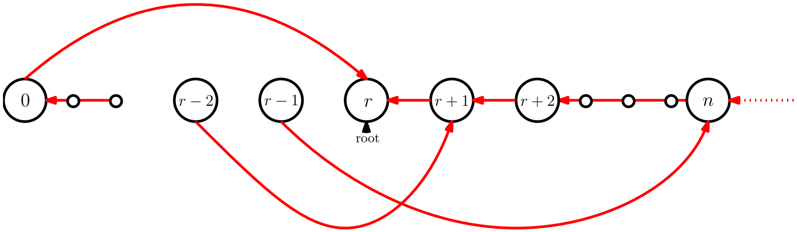


$$\mathrm{Laplacian_G} = \begin{pmatrix} 3 & -1 & 0 & -1 & 0 & -1 \\ -1 & 5 & -2 & -1 & -1 & 0 \\ 0 & -2 & 3 & 0 & -1 & 0 \\ -1 & -1 & 0 & 3 & -1 & 0 \\ 0 & -1 & -1 & -1 & 4 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$|\mathrm{ST}(G)| = \det\left(\mathrm{Laplacian}_G^{(A)}\right) = 98.$$

ST(G, r):= set of spanning trees of G **(finite graph)** rooted at r.



$W_M(T, r) := \prod_{\vec{e} \in T} M_{\vec{e}}$ with edges pointing towards the root r.

Weighted Matrix-tree theorem [Kirchhoff]

$$\sum_{T \in \mathrm{ST}(G,r)} W_M(T, r) = \det\left(\mathrm{Id} - M^{(r)}\right),$$

where $M^{(r)}$ is the matrix M deprived of the line and column r.

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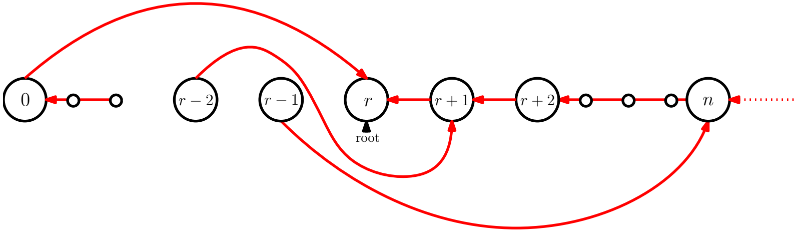

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$\text{Forest}(N, R)$ = set of spanning forests of N containing the roots in $R \subset N$. We consider each tree oriented towards its root.

Proposition (F.-Marckert'21)

Consider the graph $G = (\mathbb{N}, \{(i,j) : \mathrm{U}_{i,j} > 0\})$, *weighted by the transition matrix* U *almost-upper triangular. We have*

$$\sum_{F \in \text{Forest}(\mathbb{N}, [x, +\infty))} W_{\mathrm{U}}(F) = \det((\mathrm{Id} - \mathrm{U})_{[0,x-1]}).$$

Markov chain tree theorem

The invariant probability measure ρ of M satisfies

$$\rho_v \overset{(\text{Alg})}{=} \frac{\det(I - M^{(v)})}{Z} \overset{(\text{MTT})}{=} \frac{\sum_{T \in \text{ST}(G,v)} W_M(T, v)}{Z}$$

Proof: uniqueness and expression of the invariant measure

Uniqueness of the invariant measure: Triangular system

$$\pi_b = \sum_{a \le b+1} \pi_a \mathsf{U}_{a,b} \iff \pi_{b+1} = (\pi_b - \sum_{a \le b} \pi_a \mathsf{U}_{a,b}) / \mathsf{U}_{b+1,b}$$

Explicit expression and positive recurrence criteria:
Set $\mathsf{U}(n)$ as the matrix in $\mathbb{N} \cap [0, n]$ (chain truncated at n)

$$\begin{cases} \mathsf{U}(n)_{i,j} & = \mathsf{U}_{i,j}, \quad \text{for} \quad i \in [0, n], j \in [0, n-1] \\ \mathsf{U}(n)_{i,n} & = \sum_{j \ge n} \mathsf{U}_{i,j}. \end{cases}$$

From the Markov chain tree theorem ($U(n)$ lives on a finite state space)

$$\rho_a(n) = \alpha_n \det(\mathrm{Id} - \mathsf{U}(n)^{(a)})$$

Claim: there exist α'_n such that

$$\begin{aligned}\det(\mathrm{Id} - \mathsf{U}(n)^{(a)}) &= \det(\mathrm{Id} - \mathsf{U}_{[0,a-1]}) \prod_{j=a+1}^{n} \mathsf{U}_{a,a-1} \\ &= \alpha'_n \det(\mathrm{Id} - \mathsf{U}_{[0,a-1]}) / \prod_{j=1}^{a} \mathsf{U}_{a,a-1}.\end{aligned}$$

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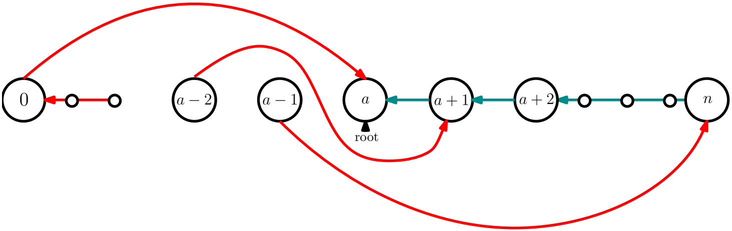

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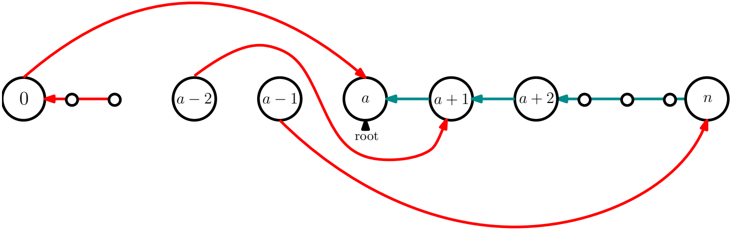

All transitions on the RHS of first line come from points in $[0, n]$ and go to points in $[0, n-1]$, in particular here $U(n)$ and U coincide.

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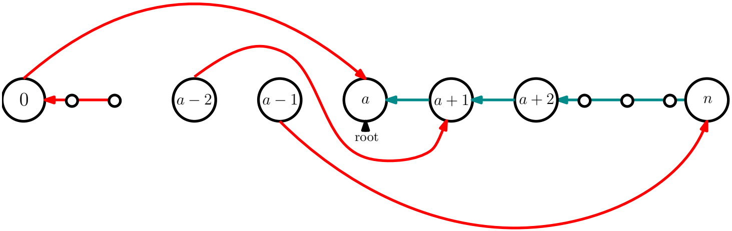

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$$\prod_{j=1}^{n} \mathrm{U}^{(n)}_{a,a-1} = \prod_{j=1}^{n} \mathrm{U}_{a,a-1}.$$

Gathering everything

$$\frac{\rho(n)_a}{\alpha_n\alpha'_n} = \frac{\det(\mathrm{Id} - \mathsf{U}_{[0,a-1]})}{\prod_{j=1}^{a} \mathsf{U}_{a,a-1}} = \pi_a$$

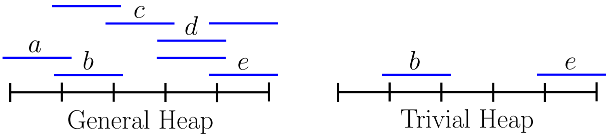
the result follows from the following lemma.

Lemma (F.-Marckert'21)

The transition matrix U *admits* π *as an invariant positive measure, if and only if there exists a sequence* $(c_n, n \geq 0)$ *such that* $c_n\rho(n) \to \pi$ *weakly.*

Combinatorial warm up II: Heaps of pieces

Informally: some "elements" that are stacked.



Formally: a set of letters $\mathcal{P}$ is given and a binary relation R:
– $x \not R y$ means that x commutes with y (that is $xy = yx$),
– xRy means that x does not commute with y.

Heap of dominos: $\mathcal{P} = \{a, b, c, d, e\}$ aRb, bRc, cRd, dRe.

Heaps: Equivalence classes of words

$w \sim w'$ if they are equal up to a finite number of allowed commutations of consecutive letters.

General heap: (left) Equivalence class of words describing the history of the stack $= baeddecb = baeddceb = ebaddbce = \ldots.$
Trivial heap: (right) All the pieces on the ground $ae = ea$

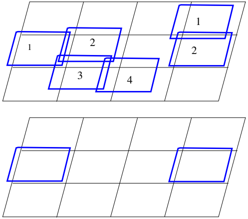


Figure: Heaps of squares. They do not commute if they share a side.

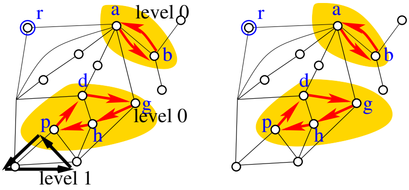


Figure: Heaps of oriented cycles. They do not conmute if they share a vertex.

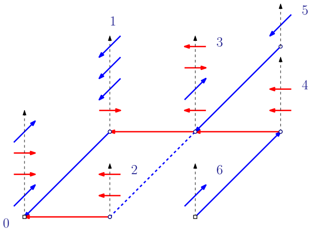


Figure: Heaps of outgoing edges. They do not commute if they start at the same point.

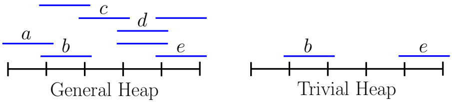


Figure: Heaps of dominoes. They do not commute if they share one extremity.

Heap of pieces

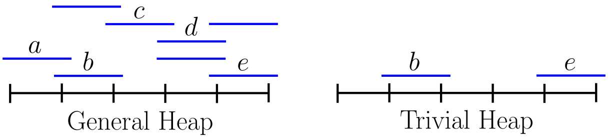
For a heap H

$$\mathit{Weight}(H) = \prod_{e \in H} w(e)$$

where $w : \mathcal{P} \to \mathbb{R}$ (or any formal commutative set)

Inversion lemma

$$\sum_{H \in \text{Heaps}} \mathit{Weight}(H) = \frac{1}{\sum_{H \in \text{TrivialHeaps}} (-1)^{|H|} \mathit{Weight}(H)}$$



Example : Weight x for each piece, $\mathit{Weight}(H) = x^{|H|}$,

$$\sum_{H \in \text{Heaps}} \mathit{Weight}(H) = \frac{1}{1 - 5x + 6x^2 - x^3}$$

In particular for the heaps of oriented cycles (HC) avoiding 0 with weights given by $M = (M_{i,j})_{i,j\in[0,n]}$

$$\sum_{\substack{HC\in\textbf{Heaps}\\ \text{avoiding } 0}} W(HC) = \left(\sum_{\substack{HC\in\textbf{TrivialHeaps}\\ \text{avoiding } 0}} (-1)^{|HC|} W_M(HC)\right)^{-1}$$
$$= \left(\sum_{C\in\mathcal{C}} (-1)^{N(C)} \prod_{c \text{ cycles of } C} \prod_{\vec{e}\in c} M_{\vec{e}}\right)^{-1}$$
$$= \det\left(I - M^{(0)}\right)^{-1}$$

where the last sum ranges over
$\mathcal{C}$ = set of collection of disjoint oriented cycles of length ≥ 1 avoiding 0.

Generalised inversion lemma

For $\mathcal{P}$ a set of pieces and $\mathcal{M} \subset \mathcal{P}$ one has

$$\sum_{\substack{H \in \mathbf{Heaps}(\mathcal{P}), \\ \text{maximal piece } \in \mathcal{M}}} Weight(H) = \frac{\sum_{H \in \text{TrivialHeaps}(\mathcal{P} \setminus \mathcal{M})} (-1)^{|H|} Weight(H)}{\sum_{H \in \text{TrivialHeaps}(\mathcal{P})} (-1)^{|H|} Weight(H)}$$

For Y a Markov chain with transition matrix U consider the hitting time of the set A as $\tau_A = \inf\{n \geq 0 : Y_n \in A\}$

$$u_b = \mathbb{P}(\tau_0(Y) \leq \tau_{[b,+\infty)}(Y) | Y_0 = 1)$$

Recurrence iff $u_b \to 1$ as $b \to \infty$.

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- Notice that $w = w_1(1,0)$ where w_1 is a path starting and ending at 1 and staying completely in $[1, b-1]$, therefore it can be seen as a heap of cycles in $[1, b-1]$ with maximal piece adjacent to 1.

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- By the generalized inversion lemma we have that

$$u_b = \frac{\det(\mathrm{Id} - \mathrm{U}_{[2,b-1]})}{\det(\mathrm{Id} - \mathrm{U}_{[1,b-1]})} \mathrm{U}_{1,0}$$

For Y a Markov chain with transition matrix U consider the hitting time of the set A as $\tau_A = \inf\{n \geq 0 : Y_n \in A\}$

$$u_b(x) = \mathbb{P}(\tau_0(Y) \leq \tau_{[b,+\infty)}(Y) | Y_0 = x)$$

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- By the generalized inversion lemma we have that

$$u_b(x) = \frac{\det(\mathrm{Id} - \mathrm{U}_{[x+1,b-1]})}{\det(\mathrm{Id} - \mathrm{U}_{[1,b-1]})} \prod_{j=1}^{x} \mathrm{U}_{j,j-1}$$

Theorem : Almost-lower triangular criteria (F.-Marckert'21)

Invariant measure

• **Finite state space** $\mathbb{N} \cap [0, s]$ the following measure π with $\pi_0 = 1$

$$\pi_a = \pi_0 \det\left(\mathsf{Id} - \mathsf{L}_{[a+1,s]}\right) \prod_{i=1}^{a} \mathsf{L}_{i-1,i} \quad \text{for all } a \in [1, s].$$

is the unique invariant by L up to a constant factor.

• **Infinite state space** $\mathbb{N}$ the Markov chain may have none, one or multiple invariant measures (we give examples of each case).

• The Markov chain is **recurrent** if and only if

$$\lim_{b \to +\infty} \frac{\prod_{j=1}^{b-1} \mathsf{L}_{j,j+1}}{\det(\mathsf{Id} - \mathsf{L}_{[1,b-1]})} = 0.$$

Connection between almost-upper and almost-lower

Proposition

Consider an irreducible ◹ *transition matrix* U, *with invariant measure* π. *Set* L *as*

$$\mathsf{L}_{i,j} = \frac{\pi_j}{\pi_i}\mathsf{U}_{j,i}.$$

Then,

- L *is an irreducible* ◺ *transition matrix on* $\mathbb{N}$, *with invariant measure* π *too.*
- L *is recurrent if and only if* U *is recurrent.*
- L *is positive recurrent if and only if* U *is positive recurrent.*

Recovering the results of Karlin & McGregor with ours

Invariant measure and positive recurrence criteria: tridiagonal case
The uniqueness of the invariant measure with $\pi_0 = 1$ gives the equality of two formulas for the invariant mesure (Karlin-McGregor and F.-Marckert)

$$\frac{\det(\mathrm{Id} - \mathsf{T}_{[0,a-1]})}{\prod_{j=1}^{a} \mathsf{T}_{j,j-1}} = \prod_{j=1}^{a} \frac{\mathsf{T}_{j-1,j}}{\mathsf{T}_{j,j-1}} \qquad \forall a \geq 1$$

then, the positive recurrence criteria is recovered.

Recurrence criteria

Set $D_{i,j} = \det(\mathrm{Id} - \mathsf{T}_{[i,j]})$, then

$$D_{i,j} = (1 - \mathsf{T}_{i,i})D_{i+1,j} - \mathsf{T}_{i,i+1}\mathsf{T}_{i+1,i}D_{i+2,j}, \tag{1}$$

and set

$$Z_{i,j} := \frac{D_{i,j}}{D_{i+1,j}\mathsf{T}_{i,i-1}}$$

and notice that our result translates into: the MC with transition matrix T is recurrent iff

$$\lim_{b\to+\infty} \mathsf{T}_{1,0}\frac{\det(\mathrm{Id} - \mathsf{T}_{[2,b-1]})}{\det(\mathrm{Id} - \mathsf{T}_{[1,b-1]})} = \lim_{b\to+\infty} \frac{1}{Z_{1,b-1}} = 1.$$

(1) rewrites

$$Z_{i,j} = \frac{(1 - \mathsf{T}_{i,i})}{T_{i,i-1}} - \frac{\frac{\mathsf{T}_{i,i+1}}{\mathsf{T}_{i,i-1}}}{Z_{i+1,j}},$$

which gives the convergents of a continued fraction.

$$Z_{1,b-1} := c_1 + \cfrac{a_2}{c_2 + \cfrac{a_3}{\ddots + \cfrac{\ddots}{c_{b-2} + \cfrac{a_{b-1}}{c_{b-1}}}}}$$

These can be solved and give that

$$Z_{1,b-1} = 1 + \left(\sum_{k=1}^{b-1} \prod_{j=1}^{k} \frac{\mathsf{T}_{j,j-1}}{\mathsf{T}_{j,j+1}} \right)^{-1}$$

So that $\lim_{b\to\infty} Z_{1,b-1} = 1$ is equivalent to Karlin & McGregor's recurrence criteria.

$$\sum_{k\geq 0} \prod_{j=1}^{k} \frac{\mathsf{T}_{j,j-1}}{\mathsf{T}_{j,j+1}} = +\infty.$$

Repair shop

The Repair shop Markov chain is defined by $X_n = (X_{n-1} - 1)_+ + Z_n$, with $(Z_i)_{i\in\mathbb{N}}$ i.i.d. family having $\mathbb{P}(Z_k = i) = a_i$, i.e. with kernel

$$A := \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\ a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\ 0 & a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & a_0 & a_1 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots \end{bmatrix}$$

and denote by m the expectation of Z_1 (i.e. $m = \sum_{i\in\mathbb{N}} ia_i$)

Criteria for the repair shop (Brémaud '13)

The Repair shop Markov chain is

- **positive recurrent** iff $m < 1$,
- **recurrent** iff $m \leq 1$.