# Bijections for tree-decorated map and applications to random maps.

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(Work in progress with Avelio Sepúlveda (Univ. Lyon 1))

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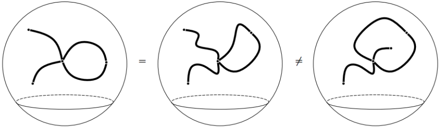
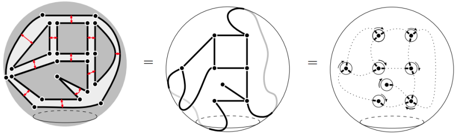
# MAPS

# Map

A **planar map** is a proper embedding of a finite connected planar graph in the sphere, considered up to direct homeomorphisms of the sphere.




Same graph, different embeddings on the sphere (sketch by N. Curien)

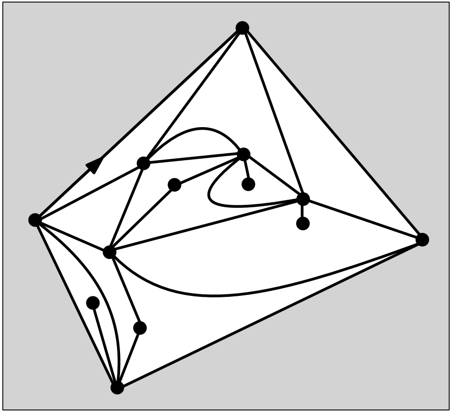



Maps seen as different objects (sketch by N. Curien)

# Map

The **faces** are the connected components of the complement of the edges. It has a distinguished half-edge: the **root edge**.

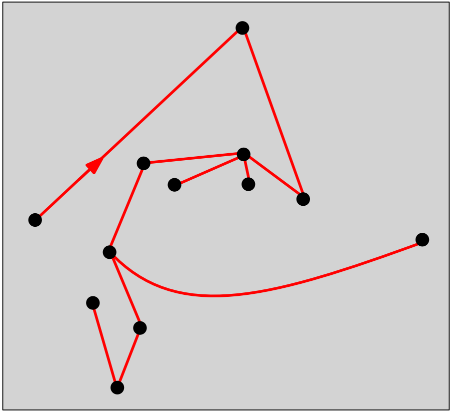
The face that is at the left of the root-edge will be called the **root-face**.



# Planar trees

A **planar tree** is a map with one face.
The set of trees with $a$ edges.

$$\mathcal{C}_a = \frac{1}{a+1}\binom{2a}{a}$$



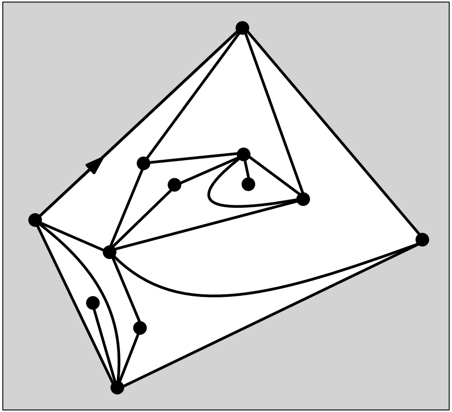
# Quadrangulations

The **degree of a face** is the number of edges adjacent to it.

A **quadrangulation** is a map whose faces have degree 4. Let $\mathcal{Q}_f$ be the set of all quadrangulations with $f$ faces, then

$$|\mathcal{Q}_f| = 3^f \frac{2}{f+2} \underbrace{\frac{1}{f+1}\binom{2f}{f}}_{\mathcal{C}_f}.$$

Analytic [Tutte '60] and Bijective [Cori-Vauquelin-Schaeffer '98].



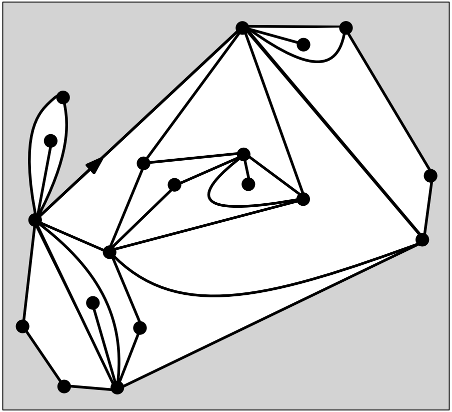
# Quadrangulations with a boundary

A quadrangulation with a boundary is a map where the **root-face** plays a special role: it has **arbitrary degree**.

The set of quadrangulations with $f$ internal faces and a boundary of size $2p$ has cardinality

$$\frac{3^f p}{(f+p+1)(f+p)} \binom{2f+p-1}{f} \binom{2p}{p}.$$

Analytic by [Bender & Canfield '94; Bouttier & Guitter '09] and bijective by [ Schaeffer '97 ; Bettinelli '15]

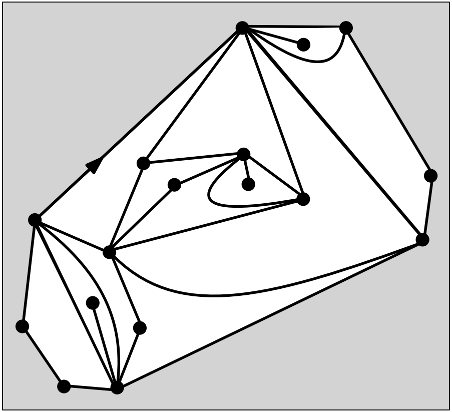


# Quadrangulations with a **simple** boundary

The set of quadrangulations with $f$ internal faces and a **simple boundary** of size $2p$ (root-face of degre $2p$) has cardinality

$$\frac{3^{f-p}2p}{(f+2p)(f+2p-1)}\binom{2f+p-1}{f-p+1}\binom{3p}{p}.$$

Analytic [Bouttier & Guitter '09]



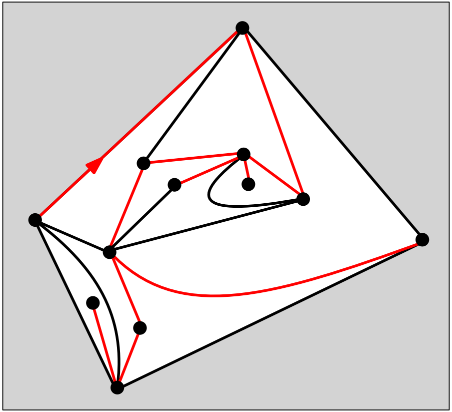
# Spanning tree-decorated maps

A spanning tree-decorated map (**ST map**) is a pair $(\mathfrak{m}, \mathfrak{t})$ where $\mathfrak{m}$ is a map and $\mathfrak{t} \subset_M \mathfrak{m}$ is a spanning tree of $\mathfrak{m}$.

The family of ST maps with $a$ edges is counted by

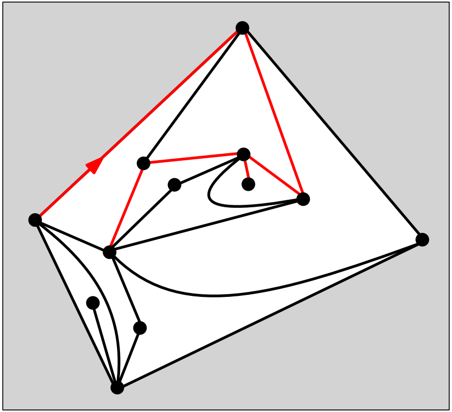
$$\mathcal{C}_a \mathcal{C}_{a+1}$$

Analytic by [Mullin '67] and bijective by [Walsh and Lehman '72; Cori, Dulucq & Viennot '86; Bernardi '06]



# ~~Spanning~~ tree-decorated maps

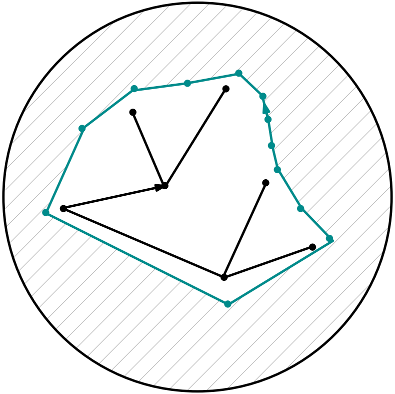
A $(f, a)$ **tree-decorated map** is a pair $(\mathfrak{m}, \mathfrak{t})$ where $\mathfrak{m}$ is a map with $f$ faces, and $\mathfrak{t}$ is a tree with $a$ edges, so that $\mathfrak{t} \subset_M \mathfrak{m}$ **containing the root-edge**.

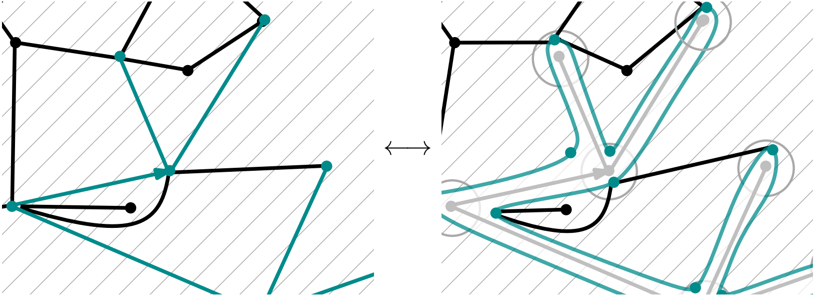


# Bijection

## Proposition (F. & Sepúlveda '19)

*The set of $(f, a)$ tree-decorated maps is in bijection with (the set of maps with a simple boundary of size $2a$ and $f$ interior faces) $\times$ (the set of trees with $a$ edges).*





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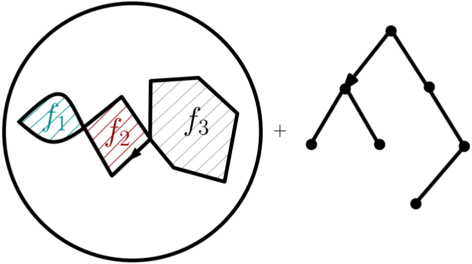

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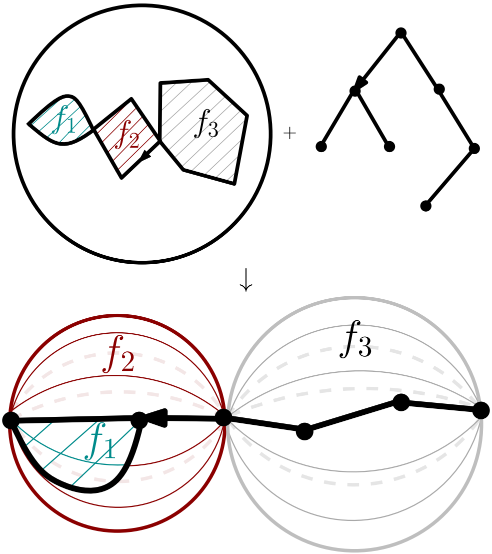

Some remarks and extensions

- From the map with a boundary the bijection preserves:
    1. Internal faces.
    2. Internal vertices.
    3. Internal edges.
- It also preserves attributes on them.
- It works with some subfamilies of trees:
    1. Binary tree- decorated Maps.
    2. SAW decorated maps (Already done by Curien & Caraceni).

# Counting results

**Corollary (F. & Sepúlveda '19)**

*The number of $(f, a)$ tree-decorated quadrangulations is*

$$3^{f-a}\frac{(2f+a-1)!}{(f+2a)!(f-a+1)!}\frac{2a}{a+1}\binom{3a}{a,a,a}$$

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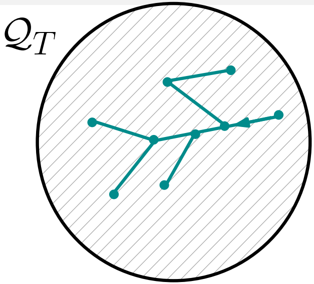
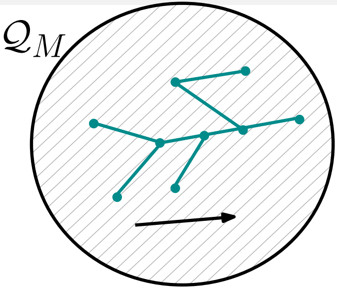
We also count

- $(f, a)$ tree-decorated triangulations.
- Maps (triangulations and quadrangulations) with a simple boundary decorated in a subtree.
- Forest-decorated maps.
- "Tree-decorated general maps".

# Re-rooting





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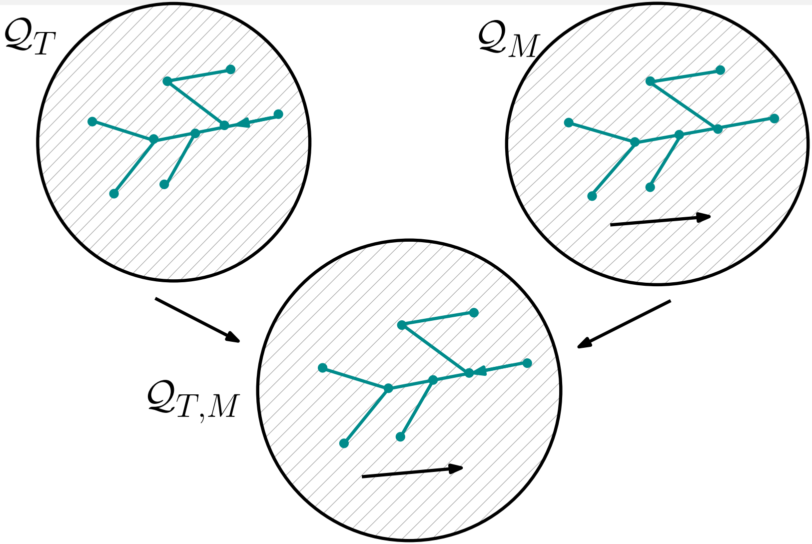

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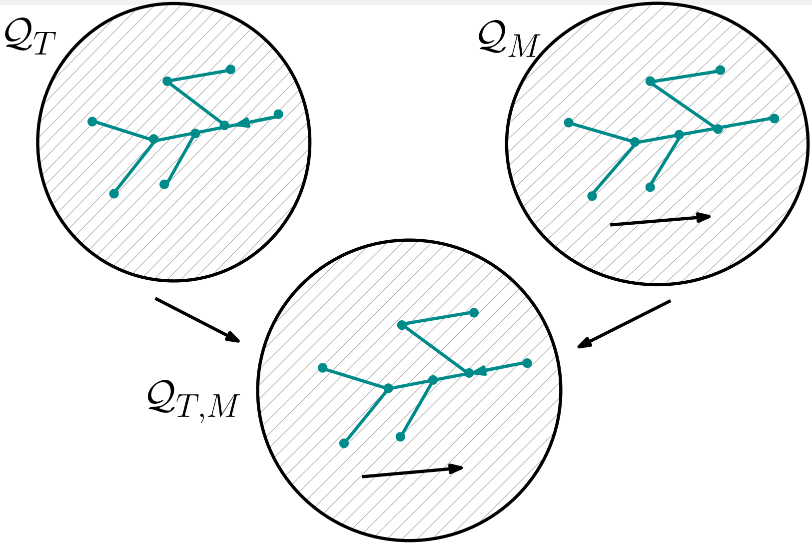


$$|\mathcal{Q}_T| \times 2|E| = |\mathcal{Q}_M| \times 2|T|$$

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A possible generalization of Catalan numbers:

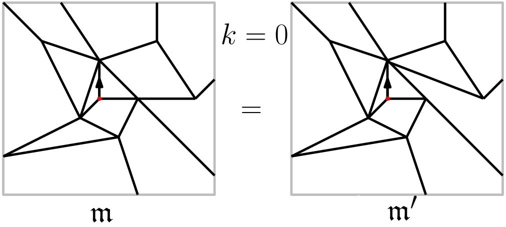
$$\mathcal{C}_{m,n} = m! \left( \prod_{i=1}^{m} \frac{1}{(n+i)} \right) \binom{(m+1)n}{\underbrace{n, n, \dots, n}_{m+1 \text{ times}}} = \binom{m+n}{n}^{-1} \binom{(m+1)n}{\underbrace{n, n, \dots, n}_{m+1 \text{ times}}}$$

# CONVERGENCE RESULTS

# Local Limits (Benjamini-Schramm Topology '01)

For a map $\mathfrak{m}$ and $r \in \mathbb{N}$, let $B_r(\mathfrak{m})$ denote the ball of radius $r$ from the root-vertex. Consider $\mathcal{M}$ a family of maps. The **local topology** on $\mathcal{M}$ is the metric space $(\mathcal{M}, \mathsf{d}_{\mathsf{loc}})$, where

$$\mathsf{d}_{\mathsf{loc}}(\mathfrak{m}_1, \mathfrak{m}_2) = (1 + \sup\{r \geq 0 : B_r(\mathfrak{m}_1) = B_r(\mathfrak{m}_2)\})^{-1}$$

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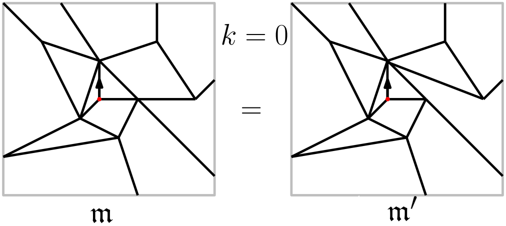


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## Proposition

*The space $(\overline{\mathcal{M}}, \mathrm{d}_{\mathrm{loc}})$ is Polish (metric, separable and complete).*

# EX 1: Uniform trees

$\mathfrak{t}_a$= Unif. tree with $a$ edges.

**Theorem (Kesten '86)**

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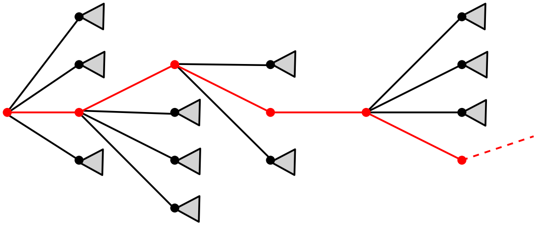
**Properties**

- $\mathfrak{t}_\infty$ *is an infinite tree.*
- *It has one infinite branch (the spine) which divides the tree in independent critical geometric Galton-Watson trees.*



$\mathfrak{t}_\infty$ **construction.**

# EX 2: Uniform quadrangulation with a boundary

$\mathfrak{q}_{f,p}$= Unif. quadrangulations with a boundary of size $2p$ and $f$ faces.

**Theorem (Curien & Miermont '12)**

$$\mathfrak{q}_{f,p} \xrightarrow[local(f\to\infty)]{(d)} \mathfrak{q}_{\infty,p} \xrightarrow[local(p\to\infty)]{(d)} UIHPQ$$

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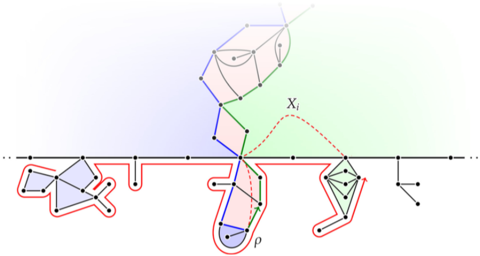
- *They also obtain the convergences for the simple boundary case.*




**UIHPQ (sketch by N. Curien & A. Caraceni).**

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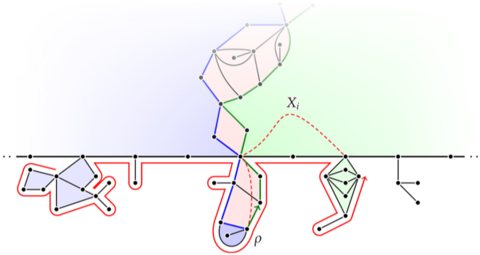


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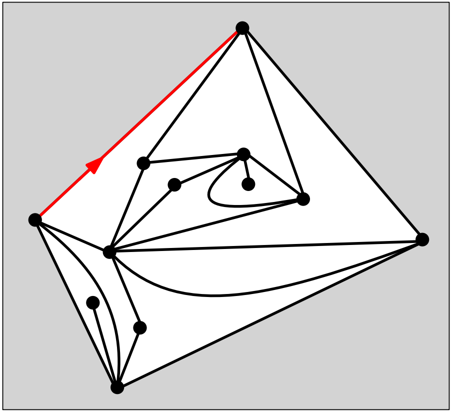
Why it is interesting to study this family??



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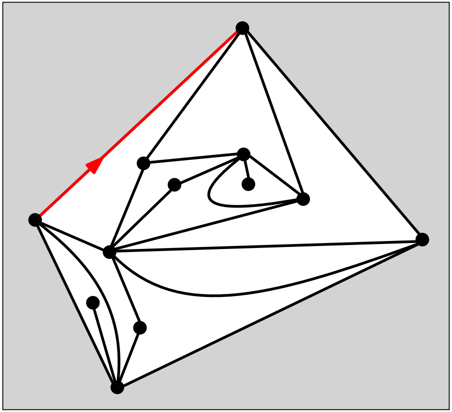
$\mathbb{P}(\mathfrak{q}_f^a = (\mathfrak{m}, \cdot)) \propto \#\{\text{trees of size } a \text{ in } \mathfrak{m}\}$



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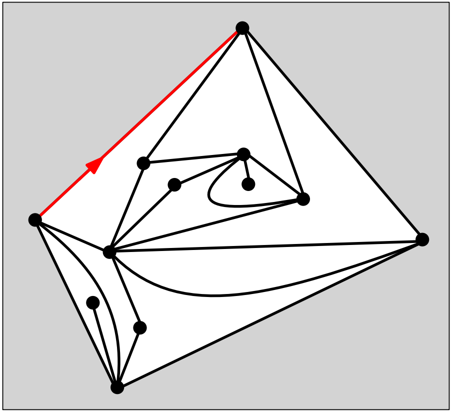
- **It interpolates**
  - $\underline{a = 1}$= Uniform quadrangulations.
  - $\underline{a = f + 1}$= Uniform ST quadrangulations.



# Local limit results

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**Proposition (F. & Sepúlveda '19+)**

*There exists a local limit for this gluing and it is the gluing root to root of* $\mathfrak{t}_\infty$ *with a* $\boldsymbol{UIHPQ}_S$*, seeing from the root.*

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## Remark

*We obtain more local limits.*

Thanks for your attention!

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# Scaling limit results

## Corollary (F. & Sepúlveda '19+)

*Let* $(\mathfrak{m}, \mathfrak{t})$ *be a Unif. tree-decorated map with* $f$ *faces and boundary of size* $a(f)$ *with* $a(f) \leq f + 1$. *Then as* $a(f) \to \infty$,

$$\left(\mathfrak{t}, \frac{\mathrm{d}_{\mathsf{Tree}}}{a(f)^{1/2}}\right) \xrightarrow[GH]{(d)} CRT.$$

# Scaling limit conjecture

## Conjecture (F. & Sepúlveda '19+)

*Let* $(\mathfrak{m}, \mathfrak{t})$ *be a Unif. tree-decorated map with* $f$ *faces and boundary of size* $a(f)$ *with* $a(f) = O(f^{\alpha})$*. Depending on* $\alpha$ *as* $f \to \infty$

$$\left((\mathfrak{m}, \mathfrak{t}), \frac{\mathrm{d}_{\mathrm{map}}}{f^{\beta}}\right) \xrightarrow[GH]{(d)} \begin{cases} \textit{Brownian map} & \textit{if } \alpha < 1/2, \beta = 1/4 \textit{(Proved)} \\ \textbf{\textit{Shocked map}} & \textit{if } \alpha = 1/2, \beta = 1/4 \textit{(In progress)} \\ \textit{Tree-decorated map} & \textit{if } \alpha > 1/2, \\ & \qquad \beta = \left(2\chi - \frac{1}{2}\right)\alpha - \chi + \frac{1}{2} \end{cases}$$

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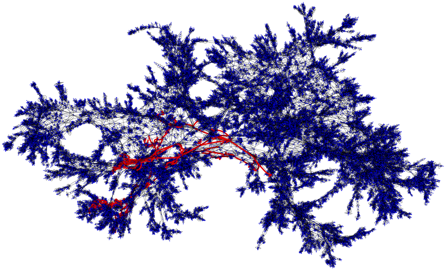

# Shocked map

Shocked map properties:

- **It is not degenerated** (Proved).
- It should be the gluing of a Brownian disk and a CRT.
- Hausdorff dim. 4 (Proved).
- The tree has Hausdorff dim. 2 (In progress, $\leq 2$ proved).
- Homeomorphic to $\mathbb{S}^2$. (Proved).

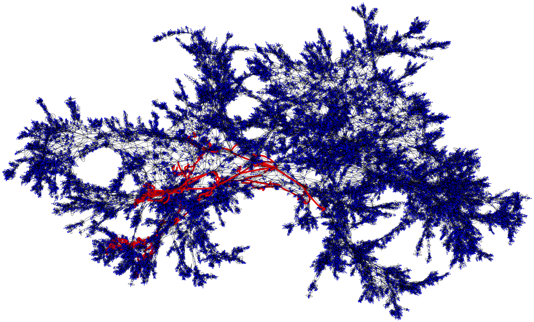

Figure: Unif. (90k,500) tree-decorated quadrangulation.

Why shocked?