Invariant measures of discrete interacting particles systems: algebraic aspects

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(joint work with J.F. Marckert).

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LaBRI

université de BORDEAUX

Particle system

Define a set of κ colors $E_\kappa := \{0, 1, \dots, \kappa - 1\}$ for $\kappa \in \{\infty, 2, 3, \dots\}$.
An **interacting particle system (IPS)** is a stochastic process $(\eta_t)_{t \in \mathbb{R}^+}$ embedded on a graph $G = (V, E)$ with configuration space in S^V. We will work with $S = E_\kappa$ and with $G = \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$. .

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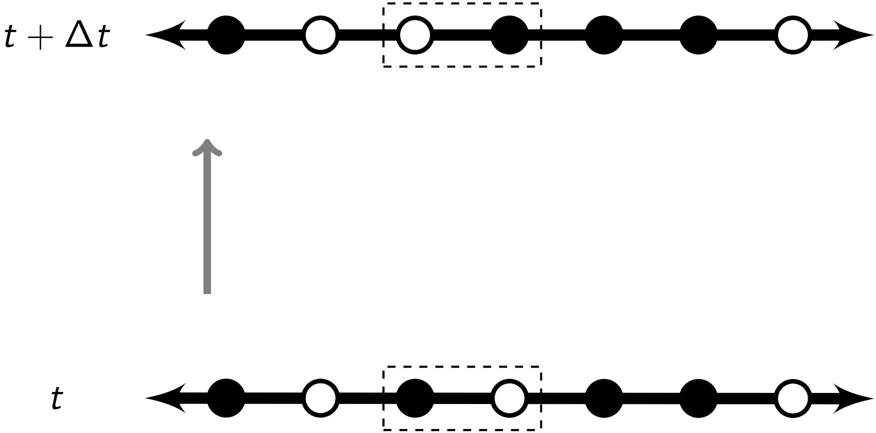
TASEP



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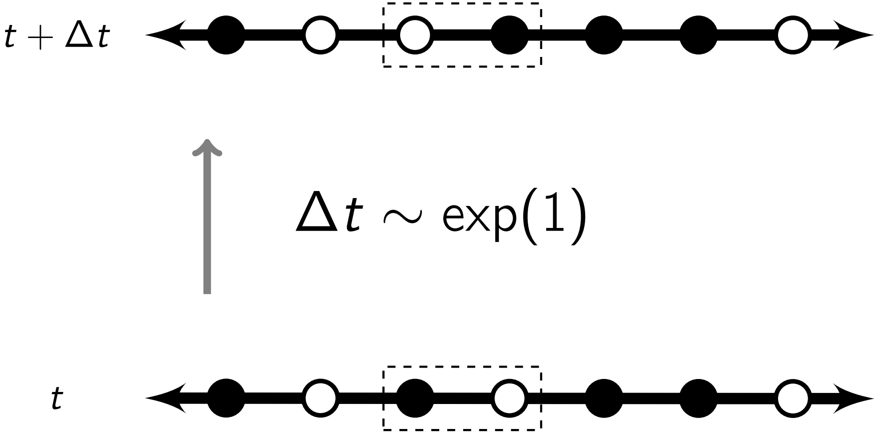
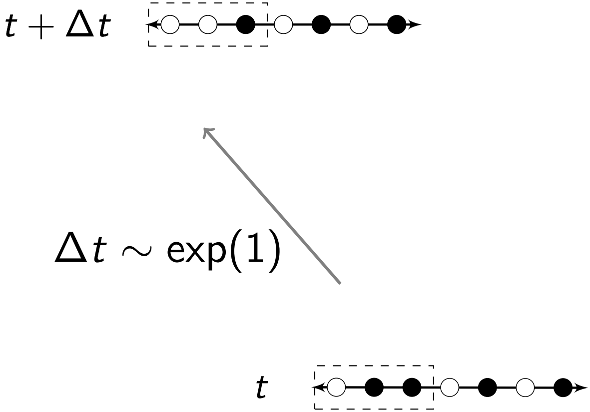
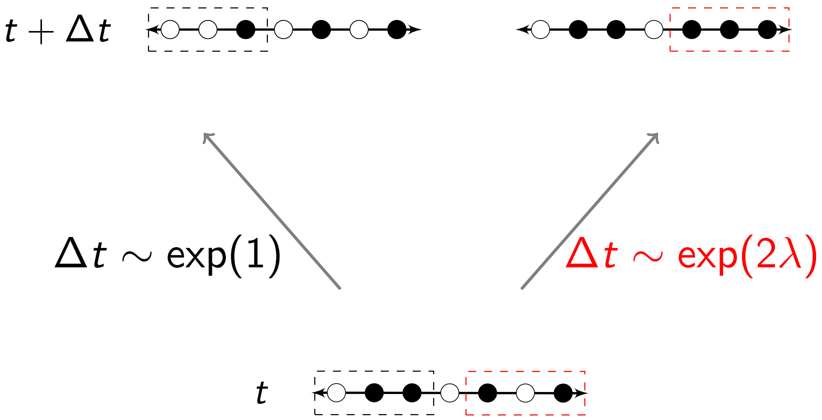
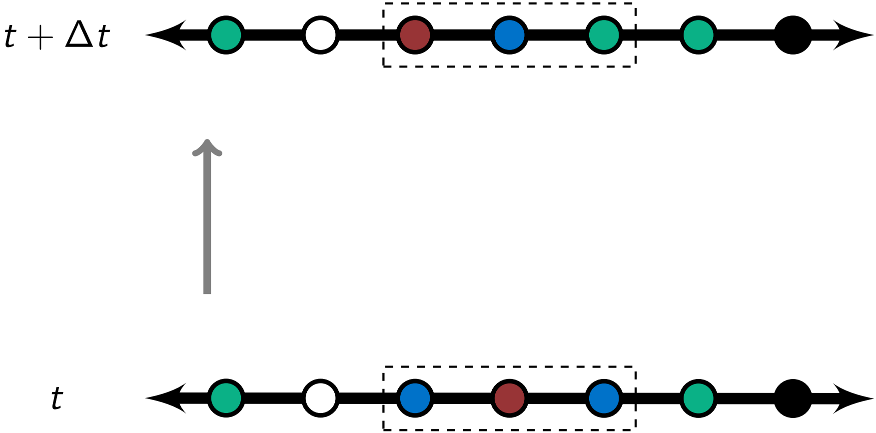



$t + \Delta t$
$\Delta t \sim \exp(1)$
t


$t + \Delta t$
$\Delta t \sim \exp(1)$
$\Delta t \sim \exp(2\lambda)$
t

General case



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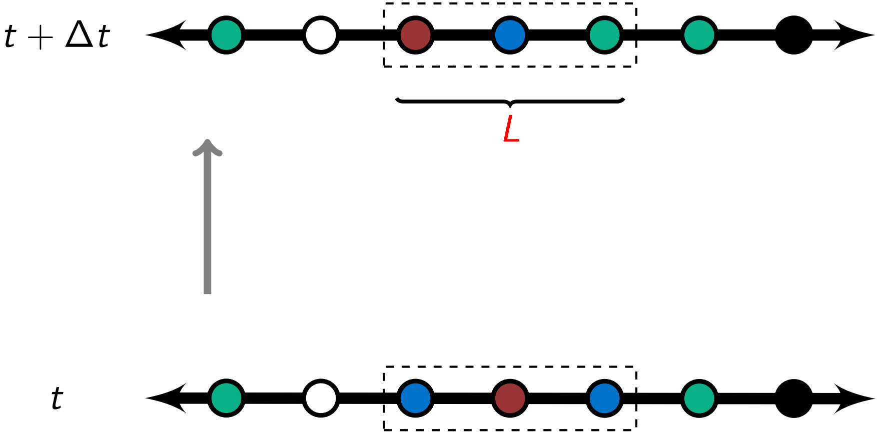

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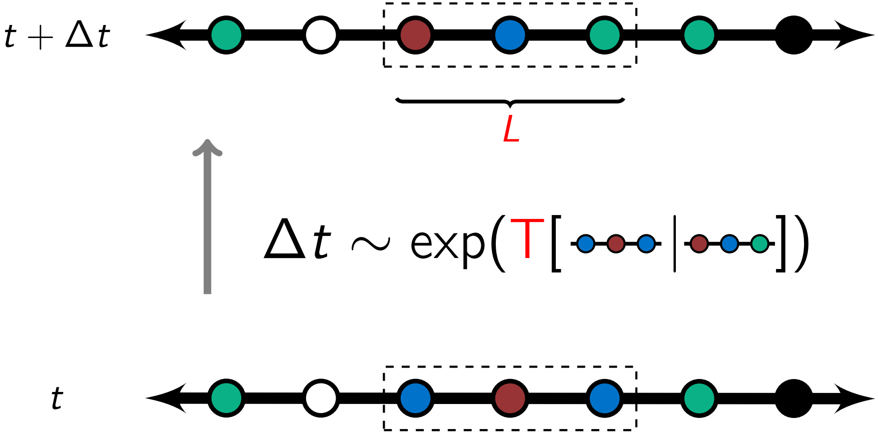

Invariant measure of particle system

Definition

A distribution μ on E_κ^V is said to be *invariant* if $\eta^t \sim \mu$ for any $t \geq 0$, when $\eta^0 \sim \mu$.

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- Simple representation? (Integrability)

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🤔 What about another type of distribution?

MARKOV!!!!!!

Consider a Markov distribution (MD) (ρ, M), with Markov Kernel (MK) M of memory $m = 1$ and ρ the invariant measure of M, i.e. for any $x \in E_\kappa^{[\![a,b]\!]}$

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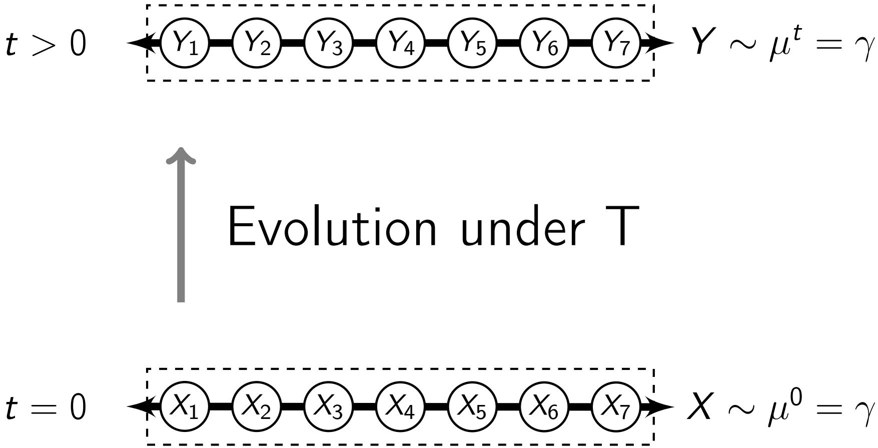
Denote by μ^t the measure of the process on $E_\kappa^{\mathbb{Z}}$ at time $t \geq 0$.



Definition

A process $(X_k, k \in \mathbb{Z}/n\mathbb{Z})$ taking its values in $E_\kappa^{\mathbb{Z}/n\mathbb{Z}}$ is said to have a Gibbs distribution $G(M)$ characterized by a MK M, if for any $x \in E_\kappa^{[\![0,n-1]\!]}$,

$$\mathbb{P}(X[\![0,n-1]\!] = x) \; = \; \frac{\prod_{j=0}^{n-1} M_{x_j, x_{j+1 \bmod n}}}{\mathrm{Trace}(M^n)} \qquad .$$

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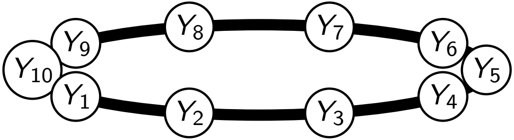
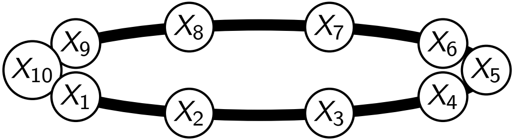
$t > 0$



$Y \sim \mu^t = \nu$

Evolution under T

$t = 0$



$X \sim \mu^0 = \nu$

Theorem 1 (F- Marckert '17)

Let E_κ be finite, $L = 2$, $m = 1$. If $M > 0$ then the following statements are equivalent for the couple (T, M):

1. (ρ, M) is invariant by T on $\mathbb{Z}$.
2. $G(M)$ is invariant by T on $\mathbb{Z}/n\mathbb{Z}$, for all $n \geq 3$
3. $G(M)$ is invariant by T on $\mathbb{Z}/7\mathbb{Z}$
4. A finite system of equations of degree 7 in M and linear in T.

Suppose μ^t is described with a MD. For any $x \in E_\kappa^{[\![1,n]\!]}$ we define

$$\mathsf{Line}_n^{M,\mathsf{T}}(x) := \frac{\partial}{\partial t}\mu^t_{[\![1,n]\!]}(x)$$

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A (ρ, M) MD under its invariant distribution is said to be AI by T on the line when $\mathrm{Line}_n \equiv 0$, for all $n \in \mathbb{N}$.

😒 But you are CHEATING!!!

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Suppose μ^t is described with a MD. For any $x \in E_\kappa^{[\![1,n]\!]}$ we define

$$\mathrm{Line}_n^{M,\mathrm{T}}(x) := \frac{\partial}{\partial t}\mu^t_{[\![1,n]\!]}(x)$$

$$= \qquad \text{Mass creation rate of } x$$

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$$\left. - \left(\rho_{x_{-1}} \prod_{k=-1}^{n+1} M_{x_k,x_{k+\mathbf{1}}}\right) \mathsf{T}^{\mathrm{out}}_{[x_j,x_{j+\mathbf{1}}]} \right)$$

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Definitions

- Define for every $a, b, c, d \in E_\kappa$

$$Z^{M,\mathsf{T}}_{a,b,c,d} := \left(\sum_{(u,v)\in E_\kappa^2} \mathsf{T}_{[u,v|b,c]} \frac{M_{a,u}M_{u,v}M_{v,d}}{M_{a,b}M_{b,c}M_{c,d}} \right) - \mathsf{T}^{\mathrm{out}}_{[b,c]}.$$

Definition

A Gibbs measure with kernel M is said to be invariant by T on $\mathbb{Z}/n\mathbb{Z}$ when $\mathrm{Cycle}_n \equiv 0$, where

$$\mathrm{Cycle}_n(x) := \sum_{j=0}^{n-1} \sum_{u,v\in E_\kappa} \left(\nu(w^j)\mathrm{T}_{[u,v|x_j,x_{j+1 \bmod n}]} - \nu(x)\mathrm{T}^{\mathrm{out}}_{[x_j,x_{j+1 \bmod n}]} \right)$$

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Extensions

Memory and amplitude

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Let E_κ be finite, $L = 2$, $m = 1$. If $M > 0$ then the following statements are equivalent:

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Theorem 1- Strongest form (F- Marckert '17)

Let E_κ be finite, $L \geq 2$, $m \in \mathbb{N}$. If $M > 0$ then the following statements are equivalent:

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2. $G(M)$ is invariant by T on $\mathbb{Z}/n\mathbb{Z}$, for all $n \geq m + L$
3. $G(M)$ is invariant by T on $\mathbb{Z}/h\mathbb{Z}$
4. A finite system of equations of degree h in M and linear in T.

$h := 4m + 2L - 1$

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- We link our results with the TASEP's matrix ansatz.
- **Problem:** MK with zero entries.

Applications

Theorem 3 (F.- Marckert '17)

Consider $\kappa < \infty$. Consider an IRM T with amplitude L, which is not identically 0.
If for infinitely many integers n the IPS with IRM T possesses an absorbing subset S_n of $E_\kappa^{\mathbb{Z}/n\mathbb{Z}}$, with $\varnothing \subsetneq S_n \subsetneq E_\kappa^{\mathbb{Z}/n\mathbb{Z}}$. Then, there does not exist any MD with any memory m with full support, invariant by T on the line.

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Corollary

The contact process do not have a MD of any memory $m \geq 0$ as invariant distribution.

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- Zero range, voter model, etc. Also when we make mild changes on these models we have some results.
- We find an IRM T which possesses some hidden Markov chain as invariant distributions. It is done using a projection from E_3 to E_2.

Thank you!